Focus is \( c^2 = b^2 - a^2 \) \( \Rightarrow \frac{1}{3} = \frac{1}{3} = \frac{\sqrt{2}}{3} \).

Note the standard form of a hyperbola is

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{with} \quad c = \sqrt{a^2 + b^2}.
\]
Remember \( q(x) = x^T A x \) for a symmetric matrix \( A \) so there is no chance of \( A \) having complex eigenvalues.

If you think about it in linear algebra language to get the quadratic form in coordinates which have no mixed terms, what we are doing is: let \( q \) be a form \( R^n \rightarrow R \) i.e.

1) Find a orthonormal basis \( \{ e_1, \ldots, e_n \} \) consisting of eigenvectors of \( A \).

2) Write \( q(x) \) in \( B \) coordinates.

Which is

\[ q^B \]

If \( x \) is in std coordinates, we want to write it in \( B \) coordinates we want

\[
[q^B] = [u_1 \ldots u_n] A [u_1 \ldots u_n]^T \]

\[ q^B = [x]_B \cdot D \cdot [x]_B \]

\[ q^B (x^B) = [x]_B^T [u_1 \ldots u_n]^T A [u_1 \ldots u_n] [x]_B \]

\[ = [x]_B^T D [x]_B \]
let $\mathbb{A} : \mathbb{C} \rightarrow \mathbb{C} = [x^2]_B$

\[
\phi(c) = \lambda_1 c_1^2 + \ldots + \lambda_n c_n^2.
\]

Consider $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

$A$ has eigenvalues $\lambda = 0$ and $\lambda = 6$.

Eigenspace for $\lambda = 0$ is $\ker A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

Eigenspace for $\lambda = 6$ is

$\ker (A - 6I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Clearly $\mathbf{v}_1 \cdot \mathbf{v}_3 = 0 = \mathbf{v}_2 \cdot \mathbf{v}_3$

But $\mathbf{v}_1, \mathbf{v}_2$ are lin. indpt but not orthogonal.

However, using Gram Schmidt we can get $\text{span} \left\{ \mathbf{u}_1, \mathbf{u}_2 \right\} = \ker (A)$
so that \( \hat{u}_1, \hat{u}_2 \) are orthonormal

and

\[ \hat{u}_2 = \frac{\hat{v}_1}{\| \hat{v}_1 \|} \]

\[ \hat{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \]

\[ \hat{u}_2 = \frac{\sqrt{2}}{\sqrt{3}} \begin{bmatrix} -1/\sqrt{2} \\ 1 \\ -1/\sqrt{2} \end{bmatrix} \]

and

\[ \hat{u}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \]

Then, in \( \theta = [\hat{u}_1, \hat{u}_2, \hat{u}_3]^T \) coordinates,

\[ g(\hat{x}) = \hat{x}^T A \hat{x} \]

\[ g(\hat{x}) = 0 \cdot c_1^2 + 0 \cdot c_2^2 + 6 \cdot c_3^2 \]

Clearly what we can say about this form is that it has minimum at \( \hat{0} \)!
If $A$ is not symmetric, this is easy. They are eigenvalues, and $\lambda_i = \lambda_j$ or $\lambda_i = \lambda_j$. Then $L(v_1) = L(v_2)$ and $L(v_2) = L(v_1)$. If $A$ is orthogonal, such that $L(v_1)$ and $L(v_2)$ are orthogonal in $\mathbb{R}^2$, then there exist $v_1, v_2$ which are orthogonal. We said that for any orthogonal transform $L: \mathbb{R}^2 \to \mathbb{R}^2$ which is linear and diagonalizable, such that $A$ is diagonal, and $L(v_1)$ and $L(v_2)$ are orthogonal in $\mathbb{R}^2$.

Remember, $g(x) = k$ is a hyperbola. Following is not on your test.

$g(x) = k$ will help you understand.
Then for any $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(x) = Ax$

if we want to find an orthogonal set $\{\hat{w}_1, \hat{w}_2\}$

to orthogonalize the set $\text{eigenbasis}$ for $A^T A$.

Now, $A^T A \hat{w}_1 = \sigma_1 \hat{w}_1$ and $\|A \hat{w}_1\|^2 = \hat{w}_1^T A^T A \hat{w}_1$

$= \sigma_1 \hat{w}_1^T \hat{w}_1$

$= \sigma_1 \|\hat{w}_1\|^2$

$= \sigma_1$

Similarly $\|A \hat{w}_2\|^2 = \sigma_2$.

Now define $\hat{u}_1 = A \hat{w}_1$, $\hat{u}_2 = \dfrac{A \hat{w}_2}{\sigma_2}$

$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$
\[ A = \sqrt{\sum \lambda_k} U \Sigma V^T \]

In general, if \( A \) is a \( m \times n \) matrix with or singular values, we will have

\[ A = U\Sigma V^T \]

where \( U \) is a \( \{u_1, \ldots, u_m\} \) is an eigen basis of \( A^+ A \).

Then the idea is that very often several of the \( \sigma_i \)'s are almost zero, by ignoring them we can get approximate values of \( A \) so if the matrix is strong some data to transfer then simplifies what needs to be sent.