Sample questions for the final

Note these do not cover all of the syllabus for the test and are only to be used as a sample.

(1) Let $A$ be a $2 \times 2$ matrix with eigenvalues 1 and 4. Let $\text{Ker } (A - I) = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ and $\text{Ker } (A - 4I) = \text{Span}\left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$.

(a) Is $A$ diagonalizable? If yes, write out the diagonalization, else explain why $A$ is not diagonalizable?

(b) Find a diagonal matrix $B$ such that $B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$.

(c) Use parts 1(a) and 1(b) to find a matrix $X$ such that $X^2 = A$.

(2) Let $A$ be the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

(a) Find the eigenvalues and eigenspaces of $A$. Write down an orthogonal basis of $\mathbb{R}^2$ consisting of eigenvectors of $A$.

(b) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of $A$.

(c) Let $T$ denote the transformation described by $A$. Write down the matrix of $T$ with respect to the new eigenbasis you wrote down in 2(a).

(d) Explain what the diagonalization of $A$ describes in terms of $T$.

(3) Solve the following system of differential equations.

\[ \frac{dx_1}{dt} = x_1(t) - 2x_2(t) \]
\[ \frac{dx_2}{dt} = 2x_1(t) + x_2(t) \]

Given $x_1(0) = 1$ and $x_2(0) = -1$. What happens to $x_1(t), x_2(t)$ as $t \to \infty$?

(4) Find all solutions in $C^\infty$ to the differential equation

\[ f'''(t) + f'(t) = e^t \]

given that $f_p(t) = e^t/2$ satisfies the equation.

(5) Answer the following in short. Give justification for your answers.

(a) Let $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 6$. Find $\det \begin{bmatrix} a + 2d & b + 2e & c + 2f \\ g & h & i \\ 2d & 2e & 2f \end{bmatrix}$.

(b) Let $V = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \right\}$. Find a basis of $V$.

(c) Give an example of a $3 \times 3$ matrix $A$ with eigenvalues 5, -1 and 3.
(d) If $A$ is a $3 \times 3$ orthogonal matrix find all possible values of its determinant.

(e) Let $A^2 = I$. Find Ker $A$.

(6) State true or false with justification.
(a) Let $A$ be a $3 \times 3$ matrix. If $Ax = 0$ has infinitely many solutions then the column vectors of $A$ span $\mathbb{R}^3$.
(b) Let $A$ be a $3 \times 3$ matrix with a set of eigenvectors spanning $\mathbb{R}^3$. Then $A$ is diagonalizable.
(c) Let $A$ be a $3 \times 3$ matrix with linearly independent column vectors. Then $A$ is diagonalizable.
(d) If $A$ is an invertible $3 \times 3$ matrix then $AB = AC$ implies $B = C$.

(7) State whether the following are subspaces of $\mathbb{R}^3$. Justify your answers.
(a) \[ \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + y = -z \right\} . \]
(b) \[ \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} . \]

(8) Write short answers to the following.

(i) Let \[
\begin{bmatrix}
1 & 3 & -1 \\
0 & -5 & 2 \\
2 & -1 & 0
\end{bmatrix}
\]
be the inverse of $A$. Find an appropriate matrix $X$ so that $XA = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}^T$. Is $X$ invertible? Why or why not?

(ii) $A$ is a diagonalizable $2 \times 2$ matrix with eigenvalues 1 and -1. Show that $A^2 = I$.

(iii) $A$ is a $n \times n$ matrix such that $AA^T = I$. What values can determinant of $A$ take?

(iv) If $\{v_1, v_2, v_3\}$ are linearly independent vectors in $\mathbb{R}^5$ and $v_4 = v_3 - v_2 + v_1$, then is $\{v_1, v_2, v_4\}$ is linearly independent? Why or why not?

(v) If $A$ has eigenvalues 1, 3 and $\frac{2}{3}$, find determinant of $A$.

(vi) If $A$ is a invertible $3 \times 3$ matrix and $v_1, v_2, v_3$ are linearly independent vectors in $\mathbb{R}^3$. Show that $Av_1, Av_2, Av_3$ are linearly independent.
(9) Let
\[ A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \]
(a) Find the orthogonal diagonalization of \( A \).
(b) Does the following equation represent an ellipse or a hyperbola.
\[ 2x_1^2 + 2x_1x_2 + 2x_2^2 = 1 \]

(10) State True or False with justification.

(i) Let \( C = AB \) for \( 4 \times 4 \) matrices \( A \) and \( B \). If \( C \) is invertible then \( A \) is invertible.

(ii) Let \( W \) be a subspace of \( \mathbb{R}^4 \) and \( v \) be a vector in \( \mathbb{R}^4 \). If \( v \in W \) and \( v \in W^\perp \) then \( v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \).

(iii) Let \( V \) be a vector space and \( W \) be a subspace of \( V \). If \( \text{Dim} \, W = \text{Dim} \, V \) then \( W = V \).

(iv) If \( A \) is a invertible \( 3 \times 3 \) matrix and \( B \) and \( C \) are \( 3 \times 3 \) matrices, then \( AB = AC \) implies \( B = C \).