Partial solutions to Sample questions for the final

Please use these only as a reference, there might be typos and errors.

(1) Let $A$ be a $2 \times 2$ matrix with eigenvalues 1 and 4. Let $\text{Ker } (A - I) = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ and $\text{Ker } (A - 4I) = \text{Span}\left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$.

(a) Is $A$ diagonalizable? If yes, write out the diagonalization, else explain why $A$ is not diagonalizable?

Hint. Since $A$ is a $2 \times 2$ matrix with 2 distinct eigenvalues, $A$ is diagonalizable. The space $\text{Ker } (A - I)$ is the eigenspace of $\lambda = 1$ and $\text{Ker } (A - 4I)$ is the eigenspace of $\lambda = 4$. So now given the eigenvectors we can diagonalize the matrix, $A = SDS^{-1}$ where $S$ consists of the eigenvectors and $D$ has its diagonal elements as the eigenvalues.

(b) Find a diagonal matrix $B$ such that $B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$.

Ans. We just need to take square root of the diagonal elements since $B$ has to be diagonal.

(c) Use parts 1(a) and 1(b) to find a matrix $X$ such that $X^2 = A$.


(2) Let $A$ be the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

(a) Find the eigenvalues and eigenspaces of $A$.

Ans. Eigenvalues are 1 and 3 with eigenspaces $\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, $\text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

(b) Find a orthogonal basis of $\mathbb{R}^2$ consisting of eigenvectors of $A$.

Ans. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

(c) Let $T$ denote the transformation described by $A$. Write down the matrix of $T$ with respect to the new eigenbasis you wrote down in 2(a).

Ans. The matrix of $T$ with respect to $\mathcal{B}$ is given by $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

(d) Explain what the diagonalization of $A$ describes in terms of $T$.

Ans. The equality $A = SDS^{-1}$ describes the relation between the transformation $T$ in standard coordinates and that in $\mathcal{B}$ coordinates.

(3) Solve the following system of differential equations.

\[
\frac{dx_1}{dt} = x_1(t) - 2x_2(t)
\]
\[
\frac{dx_2}{dt} = 2x_1(t) + x_2(t)
\]

Given $x_1(0) = 1$ and $x_2(0) = -1$. What happens to $x_1(t), x_2(t)$ as $t \to \infty$?
Ans. We find out that the matrix has eigenvalues $1 \pm 2i$ and therefore,

\[
x(t) = c_1 e^{(1+2i)t} \vec{v}_1 + c_2 e^{(1-2i)t} \vec{v}_2
\]

\[
= e^t (c_1 e^{2i(t)} \vec{v}_1 + c_2 e^{-2i(t)} \vec{v}_2)
\]

As $t \to \infty$, $x(t)$, $e^t \to \infty$. Further given the form of the matrix in the dynamical system, we can say that $x(t)$ takes values in an outward circular spiral. (you need the last bit only if the question asks for a phase portrait.)

(4) Find all solutions in $C^{\infty}$ to the differential equation

\[
f''''(t) + f'(t) = e^t
\]

given that $f_p(t) = e^t/2$ satisfies the equation.

Ans. Given the particular solution you only need to find solutions to

\[
f''''(t) + f'(t) = 0.
\]

Eigenfunction gives solution $\lambda = \pm i$. Then solutions to the differential equation are given by

\[
f(t) = c_1 \cos t + c_2 \sin t + e^t/2,
\]

for real numbers $c_1$ and $c_2$.

(5) Answer the following in short. Give justification for your answers.

(a) Let \[\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 6.\] Find \(\det \begin{bmatrix} a + 2d & b + 2e & c + 2f \\ g & h & i \\ 2d & 2e & 2f \end{bmatrix}.\)

Ans. \[\det \begin{bmatrix} a + 2d & b + 2e & c + 2f \\ g & h & i \\ 2d & 2e & 2f \end{bmatrix} = -12.\]

(b) Let $V = \text{Span}\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \}$. Find a basis of $V$. 
Ans. We need to find out which ones of these vectors are linearly independent.

\[
a_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} = \vec{0}
\]

We can see that the two pivots are 1 and 1. This shows that

\[
\begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

and

\[
\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}
\]

are linearly independent and span \(V\). Therefore, they form a basis of \(V\).

(c) Give an example of a \(3 \times 3\) matrix \(A\) with eigenvalues 5, -1 and 3.

Ans. \[
\begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}
\]

(d) If \(A\) is a \(3 \times 3\) orthogonal matrix find all possible values of its determinant.

Ans. \(\det A = \pm 1\)

(e) Let \(A^2 = I\). Find \(\text{Ker } A\).

Ans. \(A\) is invertible \(\implies\) that \(\text{Ker } A = \vec{0}\).

(6) State true or false with justification.

(a) Let \(A\) be a \(3 \times 3\) matrix. If \(Ax = 0\) has infinitely many solutions then the column vectors of \(A\) span \(\mathbb{R}^3\).

Ans. False: If \(Ax = 0\) has infinitely many solutions than \(A\) is not invertible and therefore the column vectors are not linearly independent and cannot span \(\mathbb{R}^3\).

(b) Let \(A\) be a \(3 \times 3\) matrix with a set of eigenvectors spanning \(\mathbb{R}^3\). Then \(A\) is diagonalizable.

Ans. True. If the eigenvectors span \(\mathbb{R}^3\) then there exist 3 linearly independent eigenvectors of \(A\) and therefore, \(A\) is diagonalizable.

(c) Let \(A\) be a \(3 \times 3\) matrix with linearly independent column vectors. Then \(A\) is diagonalizable.
Ans. False: For instance \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 2
\end{bmatrix}
\] is invertible but not diagonalizable.

(d) If \( A \) is an invertible \( 3 \times 3 \) matrix then \( AB = AC \) implies \( B = C \).

Ans. True. Multiply both sides by \( A^{-1} \).

(7) State whether the following are subspaces of \( \mathbb{R}^3 \). Justify your answers.

(a) \( \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y = -z, 2x - 1 = y \} \).

Ans. False: Does not have the 0 vector.

(b) \( \{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \} \).

Ans. True: check to see it has all the right properties.

(8) Write short answers to the following.

(i) Let \( \begin{bmatrix} 1 & 3 & -1 \\ 0 & -5 & 2 \\ 2 & -1 & 0 \end{bmatrix} \) be the inverse of \( A \). Find an appropriate matrix \( X \) so that \( XA = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}^T \). Is \( X \) invertible? Why or why not?

Ans. \( X = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}^T A^{-1} \). \( X \) cannot be invertible since it is not a square matrix.

(ii) \( A \) is a diagonalizable \( 2 \times 2 \) matrix with eigenvalues 1 and -1. Show that \( A^2 = I \).

Ans. Write down the diagonalization of \( A \).

(iii) \( A \) is a \( n \times n \) matrix such that \( AA^T = I \). What values can determinant of \( A \) take?

Ans. Apply determinants to the equation \( AA^T = I \).

(iv) If \( \{ v_1, v_2, v_3 \} \) are linearly independent vectors in \( \mathbb{R}^3 \) and \( v_4 = v_3 - v_2 + v_1 \), then is \( \{ v_1, v_2, v_4 \} \) is linearly independent? Why or why not?

Ans. Write down equation \( a_1 v_1 + a_2 v_2 + a_3 v_4 = 0 \) and substitute for \( v_4 \).

Use linear independence of \( v_1, v_2, v_3 \) to show that \( a_1 = a_2 = a_3 = 0 \).

(v) If \( A \) has eigenvalues 1, 3 and \( 2/3 \), find determinant of \( A \).

Ans. determinant = 2 (product of the eigenvalues).
(vi) If $A$ is a invertible $3 \times 3$ matrix and $v_1, v_2, v_3$ are linearly independent vectors in $\mathbb{R}^3$. Show that $Av_1, Av_2, Av_3$ are linearly independent.

Ans. Similar to part iv

(9) Let 

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(a) Find the orthogonal diagonalization of $A$.

Ans. Eigenvalues are 1 and 3. The corresponding eigenspaces are $\text{Span}\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \}$

and $\text{Span}\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}$. Normalize these vectors to get the orthogonal diagonalization

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

(b) Does the following equation represent an ellipse or a hyperbola.

$$2x_1^2 + 2x_1x_2 + 2x_2^2 = 1$$

Ans. We can write the above equation as

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$$

Let $$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$ Then we get

$$c_1^2 + 3c_2^2 = 1.$$ Clearly this describes an ellipse with minor, major axis along $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

(10) State True or False with justification.

(i) Let $C = AB$ for $4 \times 4$ matrices $A$ and $B$. If $C$ is invertible then $A$ is invertible.

Ans. Use determinant to observe that determinant of $A$ has to be nonzero and hence $A$ has to be invertible. True!
(ii) Let $W$ be a subspace of $\mathbb{R}^4$ and $v$ be a vector in $\mathbb{R}^4$. If $v \in W$ and $v \in W^\perp$ then $v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Ans. If $v \in W$ and $v \in W^\perp$ then $v.v = 0$ therefore $v = 0$.

(iii) Let $V$ be a vector space and $W$ be a subspace of $V$. If $\dim W = \dim V$ then $W = V$.

Ans. If $\dim W = \dim V = k$. Then $W$ has $k$ linearly independent vectors in it which span $W$. But $W \subset V$ implies these vectors are in $V$ also and are linearly independent and since $\dim V = k$ they have to span $V$. Therefore $W = V$.

(iv) If $A$ is a invertible $3 \times 3$ matrix and $B$ and $C$ are $3 \times 3$ matrices, then $AB = AC$ implies $B = C$.

Ans. Multiply by $A^{-1}$. 