• Let \( v_1, v_2, \cdots, v_n \in \mathbb{R}^m \). Is \( \text{Span}\{v_1, v_2, \cdots, v_n\} \) a subspace of \( \mathbb{R}^m \)?

• Is the set \( \overrightarrow{0} \) in \( \mathbb{R}^n \) a subspace of \( \mathbb{R}^n \)?

• Is the set \( \{(x, y) \in \mathbb{R}^2/2x + y = 0\} \) a subspace of \( \mathbb{R}^2 \)?

• Is the set \( \{(x, y) \in \mathbb{R}^2/2x + y = 1\} \) a subspace of \( \mathbb{R}^2 \)?

• Is the set \( \{(x, y, z) \in \mathbb{R}^2/x + y + z = 1, x - y + 2z = 0\} \) a subspace of \( \mathbb{R}^3 \)?
Consider the linear transformation we defined last class. Let
\[
A = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}.
\]

The transformation \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) is defined as
\[
T(\vec{x}) = A\vec{x}.
\]

We showed that the image of \( T = \text{Span}\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \} \).
Note that
\[
\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}
\]

Therefore if \( \text{Image } T = \text{Span} \{ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \} \).

Then, any vector in the Image \( T \) can be written as a linear combination of \( \vec{v}_1, \vec{v}_2 \) and \( \vec{v}_3 \).
Consider any \( \vec{w} \in \text{Image } T \). We have for some \( a_1, a_2, a_3 \in \mathbb{R} \),

\[
\vec{w} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + a_3 \vec{v}_3
\]

\[
= a_1 \text{vec } \vec{v}_1 + a_2 (\vec{v}_1 + \frac{1}{2} \vec{v}_3) + a_3 \vec{v}_3
\]

\[
= (a_1 + a_2) \vec{v}_1 + (a_3 + \frac{1}{2}a_2) \vec{v}_3
\]

This implies \( \vec{w} \in \text{Span}\{\vec{v}_1, \vec{v}_3\} \).

Further any vector in \( \text{Span}\{\vec{v}_1, \vec{v}_3\} \) is in \( \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \).

Therefore, \( \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{Span}\{\vec{v}_1, \vec{v}_3\} \).
A set of vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$ be vectors in $\mathbb{R}^n$ is said to be linearly independent. If the equation

$$a_1 \vec{v}_1 + \cdots + a_k \vec{v}_n = 0$$

has only the zero solution for $a_1, \cdots, a_k$.

Equivalently, a set of vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$ in $\mathbb{R}^n$ is said to be linearly independent if none of the vectors in the set can be written as a linear combination of each other.
Let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$ be a set of vectors in $\mathbb{R}^n$. The following are equivalent.

• The equation $a_1\vec{v}_1 + \cdots + a_k\vec{v}_n = 0$ has only the zero solution for $a_1, \cdots, a_k$.

• None of the vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$ can written as a linear combination of the others.
Let $A$ be the $n \times k$ matrix described with column vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$. Then the homogeneous system $A\vec{x} = \vec{0}$ has a unique solution.

Let $T$ be the linear transformation from $\mathbb{R}^k \rightarrow \mathbb{R}^n$ defined by $A$. Then Kernel of $T = \{ \vec{0} \}$.

Rank of $A = k$. 
Which of the following sets are linearly independent?

- \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \}

- \{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \}.

- \{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}
Let us go back to our previous example. The set \(
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
2 \\
0
\end{bmatrix}
\) is linearly independent.

Moreover, \(\text{Span}\{\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
2 \\
0
\end{bmatrix}\}\) is the XY-plane.

However, we cannot write down a set with fewer vectors which will span the XY-plane.
A set of vectors \( \{ \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p \} \) in a subspace \( W \) of \( \mathbb{R}^n \) is said to be a basis of \( W \) if every vector in \( W \) can be written uniquely as a linear combination of the vectors \( \vec{v}_1, \cdots, \vec{v}_p \). The vectors \( v_1, \cdots, v_n \) are called the basis vectors.

Equivalently, a set of vectors \( \{ \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p \} \) in a subspace \( W \) of \( \mathbb{R}^n \) forms a basis of \( W \) if the vectors are linearly independent and \( \text{Span}\{ \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p \} = W \).