Linear Algebra

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$$q(x_1, x_2) = \frac{1}{2}c_1^2 - \frac{1}{2}c_2^2.$$

In our example we see that level curves (curves of the form $q(x_1, \cdots, x_n) = k$) for the given quadratic forms will be hyperbolas with respect to the new coordinate basis.

Thus the knowledge of orthogonal diagonalization of A allows us to rewrite the quadratic form in a new set of coordinates with respect to which the quadratic form is easier to understand.

Now we will describe another application of symmetric matrices by using the idea that given any matrix A (does not even have to be a square matrix), $A^T A$ is a symmetric matrix.

Remember the example that you solved in your homework which showed that an invertible transformation from $\mathbb{R}^2 \to \mathbb{R}^2$ maps a unit circle onto a ellipse. This follows from the fact that a invertible transformation described by A maps some pair of orthogonal vectors on the unit circle onto orthogonal vectors.

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Then

$$(A\vec{v}_1)^T(A\vec{v}_2) = \vec{v}_1^T A^T A\vec{v}_2 = \lambda_2 \vec{v}_1^T \vec{v}_2 = 0.$$

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This allows us to prove that any invertible linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$ maps a circle into an ellipse.

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Then $A^T A$ is a symmetric $m \times m$ matrix. Note that if λ is an eigenvalue of $A^T A$ with a corresponding eigenvector $\vec{v}$ then

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This implies that $lambda \geq 0$. The eigenvalues of $A^T A$ are all positive real.
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Then by previous argument $A\vec{v}_1, \cdots, A\vec{v}_m$ are orthogonal to each other, (Note some of them may be zero since the transformation is not invertible) such that $\|A\vec{v}_i\| = \sigma_i$ for all $i = 1, \cdots, m$.
This is because

$$(A\vec{v}_i)^T (A\vec{v}_i) = \vec{v}_i^T A^T A\vec{v}_i = \lambda_i \vec{v}_i^T \vec{v}_i = \lambda_i.$$

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Let A have rank r. Then the set $\{A\vec{v}_1, A\vec{v}_2, \cdots A\vec{v}_m\}$ can have at most r non-zero vectors, since being orthogonal implies the set is linearly independent.

Then $A\vec{v}_i = \sigma_i \vec{v}_i$ and σ_i's are in descending order implies that $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_m = 0$.

Therefore, $A\vec{v}_{r+1} = \cdots = A\vec{v}_m = \vec{0}$ implies that $\{\vec{v}_{r+1}, \cdots, \vec{v}_m\} \subset \text{Ker } A$.

By rank-nulity theorem we know that Dim Ker $A = m - r$. Therefore the orthonormal set $\{\vec{v}_{r+1}, \cdots, \vec{v}_m\}$ is a basis of Ker A.

This implies that $\{A\vec{v}_1, \cdots, A\vec{v}_r\}$ is a orthogonal set of non-zero vectors (otherwise we would have a bigger linearly independent subset in Ker A) and hence linearly independent.

Since Rank of $A = r$ then $\{A\vec{v}_1, \cdots, A\vec{v}_r\}$ is a basis of Image of A.

Define

$$\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i \in \mathbb{R}^n \implies \vec{u}_i \sigma_i = A\vec{v}_i$$

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This gives us a orthonormal subset of $\mathbb{R}^n$. We can extend $\{\vec{u}_1, \cdots, \vec{u}_r\}$ to an orthonormal basis $\{\vec{u}_1, \cdots, \vec{u}_n\}$ of $\mathbb{R}^n$.

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Then we can write

$$A[\vec{v}_1 \cdots \vec{v}_m] = [\vec{u}_1 \cdots \vec{u}_n] \begin{bmatrix} \sigma_1 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \sigma_r & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{n \times m} .$$

This is called the singular value decomposition of $A = U\Sigma V^T$.

Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

Then $A^T A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

The eigenvalues of $A^T A$ are 0, 1 , 3 and, $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

Then $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a orthogonal basis of $\mathbb{R}^3$ and $\left\{ \vec{v}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a orthonormal basis of $\mathbb{R}^3$

The singular values of A are $\sigma_1 = \sqrt{3}$, $\sigma_2 = 1$ and $\sigma = 0$. Then define

$$\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i,$$

where $i = 1, 2$.

Therefore, $\vec{u}_1 = \frac{1}{3\sqrt{2}} \begin{bmatrix} -3 \\ 3 \end{bmatrix}$ and $\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

This is clearly a orthonormal basis of $\mathbb{R}^2$. Thus the singular value decomposition of A is given as follows:

$$A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T.$$