Linear Algebra

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February 13, 2009
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- multiplying a row by a scalar

For example, 
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\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
$$

is an elementary $2 \times 2$ matrix obtained by exchanging rows.

The row reduced form of a $m \times n$ matrix $A$ can then be written as $\text{rref } A = E_1 E_2 \cdots E_k$ where $E_i$ are $m \times m$ elementary matrices.
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where \( E_i \) are \( m \times m \) elementary matrices.
We described the Image of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ as

$$\text{Image } T = \{ \vec{b} \in \mathbb{R}^n : T(\vec{x}) = \vec{b} \text{ for some } \vec{x} \in \mathbb{R}^m \}$$
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Let the transformation be defined by a $n \times m$ matrix $A$, that is,

$T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^m$, and $A = [\vec{v}_1 \cdots \vec{v}_m]$, for $\vec{v}_1, \cdots, \vec{v}_m \in \mathbb{R}^n$.

Then Image $T = \text{Span } \{ \vec{v}_1, \cdots, \vec{v}_m \}$. 
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In fact, if we know all the possible solutions of $A\vec{x} = \vec{0}$, the solutions of any consistent system $A\vec{x} = \vec{b}$ is a translation of the former set of solutions.
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In fact, if we know all the possible solutions of $A\vec{x} = \vec{0}$, the solutions of any consistent system $A\vec{x} = \vec{b}$ is a translation of the former set of solutions.

For example...
Let $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a linear transformation defined by a $n \times m$ matrix $A$. 
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Namely, $T(\vec{x}) = A\vec{x}$.

Then the Kernel of $T$ is the set of vectors $\vec{x} \in \mathbb{R}^m$ such that $T(\vec{x}) = \vec{0}$.
• The vector $\vec{0} \in \mathbb{R}^n$ is always in the Kernel.
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Consider a set $V$ on which addition of elements of $V$ and scalar multiplication by $\mathbb{R}$ is well defined.
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More specifically, $u, v \in V$ implies $u + v \in V$ and $u \in V$, $c \in \mathbb{R}$ implies $cu \in V$. 
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More specifically, $u, v \in V$ implies $u + v \in V$ and $u \in V, c \in \mathbb{R}$ implies $cu \in V$.

The set is $V$ is said to be a (real) vector space if it satisfies the following properties:
Let $u, v, w \in V$ and $c, k \in R$.

- $(u + v) + w = u + (v + w)$
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- $(u + v) + w = u + (v + w)$
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- There exists a unique element denoted by $0$ in $V$ such that $u + 0 = u = 0 + u$ for all $u \in V$. 
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- \(u + v = v + u\)
- There exists a unique element denoted by \(0\) in \(V\) such that \(u + 0 = u = 0 + u\) for all \(u \in V\).
- For every \(u\) there is an unique additive inverse denoted as \((-u) \in V\) such that \(u + (-u) = 0 = (-u) + u\).
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- There exists a unique element denoted by $0$ in $V$ such that $u + 0 = u = 0 + u$ for all $u \in V$.
- For every $u$ there is an unique additive inverse denoted as $(−u) \in V$ such that $u + (−u) = 0 = (−u) + u$.
- $c(u + w) = cu + cw$
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- $(c + k)u = cu + ku$
- $1u = u$