12 March, 2010

MATH 212 Homework 4
Due on March 26, 2010

(1) Let $V$ be a finite dimensional vector space over a field $F$ with a basis $\mathcal{B} = \{v_1, v_2, \ldots, v_k\}$. Let $T : V \to V$ be a linear transformation and $A = [T]_{\mathcal{B}}^{\mathcal{B}}$. Show that the following statements are equivalent.
(a) The linear transformation $T$ is invertible.
(b) Span\{\(T(v_1), T(v_2), \ldots, T(v_k)\)\} = $V$.
(c) Dimension of $R(T) = k$.
(d) Dimension of $N(T) = 0$.
(e) $T$ is one-one.
(f) $T(v_1), T(v_2), \ldots, T(v_k)$ are linearly independent.
(g) $Ax = 0$ has a unique solution.
(h) The columns of $A$ form an linearly independent set of $F^k$.
(i) $Ax = b$ has a solution for all $b \in F^k$.
(j) Rank of $A = k$.
(k) $A$ is invertible.
(l) $A^t$ is invertible.
(m) Rows of $A$ form a linearly independent subset of $F^k$.

(2) Show that for any $A$ and $B$ are $m \times n$ matrix with entries in a field $F$ if $Ax = Bx$ for all $n \times 1$ matrices $x$ in $F$ then $A = B$.

(3) Let $T : V \to V$ be a linear transformation, where $V$ is a finite dimensional vector space over a field $F$.
(a) If $\text{Rank}(T) = \text{Rank}(T^2)$ then $R(T) \cap N(T) = \{0\}$.
(b) Show that there is some integer $k$ for which $\text{Rank}(T^k) = \text{Rank}(T^{k+1})$.
(c) Show that $V = R(T^k) \oplus N(T^k)$ for some integer $k > 0$.

(4) Let $V$ be a finite dimensional vector space over a field $F$ with a basis $\mathcal{B} = \{v_1, v_2, \ldots, v_k\}$.
(a) Let $T : V \to V$ be the transformation defined by $T(v_i) = v_{i+1}$ for all $i = 1, 2, \ldots, k-1$ and $T(v_k) = 0$.
Find the matrix $A$ representing $T$ with respect to the basis $\mathcal{B}$.
(b) Prove that $T^k = 0$ but $T^{k-1} \neq 0$.
(c) Show that for any transformation $L : V \to V$ such that $L^k = 0$ but $L^{k-1} \neq 0$, there exists an ordered basis $\mathcal{B}'$ such that $[L]_{\mathcal{B}}^{\mathcal{B}'} = A$.
(d) Prove that if $M$ and $N$ are $k \times k$ matrices such that $N^k = 0 = M^k$ but $M^{k-1} \neq 0 \neq N^k$ then $M$ and $N$ are similar.


(6) Exercise 3(c) page 116.
(7) Exercise 6(b), page 117.

(8) In \( P_2(\mathbb{R}) \) find the change of basis matrix from the basis \( \mathcal{B}_1 = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\} \) to the basis \( \mathcal{B}_2 = \{1, t, t^2\} \). Then find the the column matrix representing the polynomial \( f(t) = 1 - 2t \) in \( \mathcal{B}_1 \) coordinates.