Take home test I  
Math 413  
Due on March 24, 2010

(1) Show that none of these spaces are homeomorphic to each other.  
(a) $S^1$ with subspace topology in $\mathbb{R}^2$.  
(b) $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ with subspace topology.  
(c) $S^1 \vee S^1 = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2+y^2 = 1\} \cup \{(x, y) \in \mathbb{R}^2 \mid (x+1)^2+y^2 = 1\}$

(2) Prove that the complement of $\mathbb{Q}^2$ in $\mathbb{R}^2$ is connected.

(3) Let $(X, d)$ be a metric space and $Y$ be a subset of $X$. Let $\overline{d}$ denote the restriction of $d : X \times X \to \mathbb{R}$ to $Y \times Y$.  
(a) Verify that $\overline{d}$ defines a metric on $Y$.  
(b) Show that the metric topology on $(Y, \overline{d})$ is equivalent to the subspace topology on $Y$.

(4) Prove that the usual topology on $\mathbb{R}^2$ is strictly weaker than the topology induced on $\mathbb{R}^2$ by lexicographic (dictionary) order.

(5) Let $R, S$ be equivalence relations on a space $X$ such that $R$ is finer than $S$, that is, for $x, y \in X$; $xRy \implies xSy$. Let $p : X/R \to X/S$ be the natural function which takes the equivalence class of $x \in X$ with respect to $R$ to the equivalence class of $x$ with respect to $S$. Prove that $p$ is a quotient map.

(6) Let $X, Y$ be topological spaces and $f : X \to Y$ be a homeomorphism. Let $R$ be an equivalence relation on $X$. Then define a relation $S$ on $Y$ as $y_1S_y_2$ iff $f^{-1}(y_1)Rf^{-1}(y_2)$.  
(a) Show that this defines an equivalence relation on $Y$.  
(b) Show that the quotient spaces $X/R$ and $Y/S$ are homeomorphic.

(7) Let $X$ be a space defined to be the disjoint union of two discs $D_1 = \{(x, y) \in \mathbb{R}^2 \mid (x-\frac{3}{2})^2+y^2 \leq 1\}$ and $D_2 = \{(x, y) \in \mathbb{R}^2 \mid (x+\frac{3}{2})^2+y^2 \leq 1\}$ in $\mathbb{R}^2$. Define a relation $\sim$ on the space by identifying the points on the boundary of $D_1$ and $D_2$ as follows:  
$(a, b) \in D_1 \sim (-a, b) \in D_2$ if and only if $(a-\frac{3}{2})^2+b^2 = 1$.  
Show that the quotient space obtained by considering the partition set is homeomorphic to $S^2 \subset \mathbb{R}^3$. You can assume that $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is the quotient space obtained by identifying the boundary of a disc $D^2$ in $\mathbb{R}^2$ to a point.
(8) Let \( p : \mathbb{R} \to S^1 \) be the map \( p(t) = (cos 2\pi t, \sin 2\pi t) \). Show that if \( id_{S^1} : S^1 \to S^1 \) is the identity function then there does not exist a continuous map \( f : S^1 \to \mathbb{R} \) such that \( p \circ f = id_{S^1} \).

**Definition 0.1.** If a space \( X \) has a countable basis for its topology then \( X \) is said to be second-countable. For example, \( \mathbb{R} \) has a countable basis which consists of open intervals with rational endpoints.

It follows easily that if \( X \) is second countable then any subspace \( Y \) of \( X \) is also second countable.

(9) Prove that if \( (X_i, T_i) \) are second countable topological spaces for \( i = 1, \ldots, k \) then \( X_1 \times X_2 \times \cdots \times X_k \) with the product topology is also second countable.

(10) Let \( \mathbb{R}_l \) denote \( \mathbb{R} \) with semi-open interval topology (that is, the topology generated by sets of the form \([x, y)\) for \( x, y \in \mathbb{R} \)). Let \( L = \{(x, -x) \in \mathbb{R}^2\} \). Prove that the subspace topology on \( L \) induced by product topology on \( \mathbb{R}_l \times \mathbb{R}_l \) is discrete topology.

(11) Prove that a discrete space is second countable if and only if the underlying set is countable.

(12) Using the previous exercises, show that \( \mathbb{R}_l \), that is, real line with semi-open interval topology is not second countable.