Answers to problems on the 1996 Final Exam

1a. \[
\begin{pmatrix}
-19/2 & 59/2 & -3/2 \\
5/2 & -15/2 & 1/2 \\
-1 & 3 & 0
\end{pmatrix}
\]

2a. \[\|X + Y\|^2 + \|X - Y\|^2 = 2\|X\|^2 + 2\|Y\|^2\]

3. \((0, -3/2, 3/2, 0)\)

4a. \(\frac{1}{3}\sqrt{5}\)

4b. \[z = \frac{2}{3}x + \frac{1}{3}y + \log 3 - \frac{4}{3}\]

5. \[.9 + .3x + 1.5x^2\]

6a. \((-1/3, 0)\)

6b. \(2/9\)

7b. rank = 2, nullity = 1

Solutions to problems 6 and 7a are on pp. 2–3.
6. Let \( H(x,y) = G(x,y, f(x,y)) \).

a) Since \( H(x,y) = 0 \)
\[
0 = \frac{\partial H}{\partial x} = D_1 G(x,y, f(x,y)) + D_2 G(x,y, f(x,y)) \frac{\partial f}{\partial x}(x,y).
\]
\[
0 = \frac{\partial H}{\partial y} = D_2 G(x,y, f(x,y)) + D_3 G(x,y, f(x,y)) \frac{\partial f}{\partial y}(x,y).
\]

\((*)\)
\[
0 = f(x,y)^3 + \sqrt{1+x^2} + (3x f(x,y)^2 + y) \frac{\partial f}{\partial x}(x,y)
\]
\[
0 = \frac{\partial H}{\partial y} = D_2 G(x,y, f(x,y)) + D_3 G(x,y, f(x,y)) \frac{\partial f}{\partial y}(x,y).
\]

Substitute \( x = 2, y = 0 \), \( f(x,y) = f(2,0) = -1 \):
\[
0 = -1 + \sqrt{1+4} + (6+0) \frac{\partial f}{\partial x}(2,0) = 2 + 6 \frac{\partial f}{\partial x}(2,0)
\]
\[
\frac{\partial f}{\partial x}(2,0) = -\frac{1}{3}.
\]
\[
0 = \frac{\partial H}{\partial y} = f(x,y) + e^{x^2} + (3x f(x,y)^2 + y) \frac{\partial f}{\partial y}(x,y) \bigg|_{x=2, y=0} = 6 \frac{\partial f}{\partial y}(2,0)
\]
\[
\nabla f(2,0) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \bigg|_{x=2, y=0} = (-\frac{1}{3}, 0).
\]

b) Differentiate \((*)\) with respect to \( x \):
\[
0 = 3f^2 \frac{\partial f}{\partial x} + \frac{3}{2} x^2 (1+x^2)^{-\frac{1}{2}} + (3x f^2 + y) \frac{\partial^2 f}{\partial x^2}
\]
\[
+ (3f^2 + 6xf \frac{\partial f}{\partial x}) \frac{\partial f}{\partial x}.
\]

Substitute \( x = 2, y = 0 \), \( f = -1 \), \( \frac{\partial f}{\partial x} = -\frac{1}{3} \):
\[
0 = 3 \cdot (-1)^2 + \frac{3}{2} \cdot 4 \cdot \frac{1}{3} + (3 \cdot 2^2 + 0) \frac{\partial^2 f}{\partial x^2}(2,0)
\]
\[
+ (3(-1) + 6 \cdot 2 \cdot (-\frac{1}{3})) \left( \frac{\partial f}{\partial x} \right) (2,0)
\]
\[
= -1 + 2 + 6 \frac{\partial^2 f}{\partial x^2}(2,0).\]
\[
\frac{\partial^2 f}{\partial x^2}(2,0) = \frac{2}{9}.
\]
7 a) Since \( A^2 \neq 0 \), there exists \( x \in \mathbb{R}^3 \) such that \( A^2x \neq 0 \).

Suppose \( c_1x + c_2Ax + c_3A^2x = 0 \).

Since \( A^3 = 0 \), we also have \( A^4 = AA^3 = 0 \), and

\[
0 = A^2(c_1x + c_2Ax + c_3A^2x) = c_1A^2x + c_2A^3x + c_3A^4x = c_1A^2x
\]

\[
\therefore c_1 = 0.
\]

\[
0 = A(c_2Ax + c_3A^2x) = c_2A^2x + 0
\]

\[
\therefore c_2 = 0
\]

\[
0 = c_3A^2x
\]

\[
\therefore c_3 = 0
\]

\[
\therefore \{x, Ax, A^2x\} \text{ is linearly independent}
\]

\[
\dim \mathbb{R}^3 = 3
\]

Since any set of 3 independent elements in \( \mathbb{R}^3 \) is a basis of \( \mathbb{R}^3 \) (Theorem 1.7b), it follows that \( \{x, Ax, A^2x\} \) is a basis for \( \mathbb{R}^3 \).