E AND APPLIED MATHEMATICS

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Real Analysis

Modern Techniques and Their Applications

Second Edition

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A Wiley-Interscience Publication
JOHN WILEY & SONS, INC.
New York / Chichest...

It is trivial to verify that the intersection of any family of σ-algebras on X is again a σ-algebra. It follows that if $\mathcal{E}$ is any susbset of $\mathcal{P}(X)$, there is a unique smallest σ-algebra $\mathcal{M}(\mathcal{E})$ containing $\mathcal{E}$, namely, the intersection of all σ-algebras containing $\mathcal{E}$. (There is always at least one such, namely, $\mathcal{P}(X)$.) $\mathcal{M}(\mathcal{E})$ is called the σ-algebra **generated** by $\mathcal{E}$. The following observation is often useful:

1.1 Lemma. *If $\mathcal{E} \subset \mathcal{M}(\mathcal{F})$ then $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F})$.*

Proof. $\mathcal{M}(\mathcal{F})$ is a σ-algebra containing $\mathcal{E}$; it therefore contains $\mathcal{M}(\mathcal{E})$. ∎

If X is any metric space, or more generally any topological space (see Chapter 4), the σ-algebra generated by the family of open sets in X (or, equivalently, by the family of closed sets in X) is called the **Borel σ-algebra** on X and is denoted by $\mathcal{B}_X$. Its members are called **Borel sets.** $\mathcal{B}_X$ thus includes open sets, closed sets, countable intersections of open sets, countable unions of closed sets, and so forth.

There is a standard terminology for the levels in this hierarchy. A countable intersection of open sets is called a $\boldsymbol{G_\delta}$ set; a countable union of closed sets is called an F_σ set; a countable union of G_δ sets is called a $\boldsymbol{G_{\delta\sigma}}$ set; a countable intersection of F_σ sets is called an $\boldsymbol{F_{\sigma\delta}}$ set; and so forth. (δ and σ stand for the German *Durchschnitt* and *Summe*, that is, intersection and union.)

The Borel σ-algebra on $\mathbb{R}$ will play a fundamental role in what follows. For future reference we note that it can be generated in a number of different ways:

1.2 Proposition. *$\mathcal{B}_{\mathbb{R}}$ is generated by each of the following:*

a. the open intervals: $\mathcal{E}_1 = \{(a,b) : a < b\}$,
b. the closed intervals: $\mathcal{E}_2 = \{[a,b] : a < b\}$,
c. the half-open intervals: $\mathcal{E}_3 = \{(a,b] : a < b\}$ or $\mathcal{E}_4 = \{[a,b) : a < b\}$,
d. the open rays: $\mathcal{E}_5 = \{(a,\infty) : a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty,a) : a \in \mathbb{R}\}$,
e. the closed rays: $\mathcal{E}_7 = \{[a,\infty) : a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty,a] : a \in \mathbb{R}\}$.

Proof. The elements of $\mathcal{E}_j$ for $j \neq 3, 4$ are open or closed, and the elements of $\mathcal{E}_3$ and $\mathcal{E}_4$ are G_δ sets — for example, $(a,b] = \bigcap_1^\infty (a, b+n^{-1})$. All of these are Borel sets, so by Lemma 1.1, $\mathcal{M}(\mathcal{E}_j) \subset \mathcal{B}_{\mathbb{R}}$ for all j. On the other hand, every open set in $\mathbb{R}$ is a countable union of open intervals, so by Lemma 1.1 again, $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_1)$. That $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_j)$ for $j \geq 2$ can now be established by showing that all open intervals lie in $\mathcal{M}(\mathcal{E}_j)$ and applying Lemma 1.1. For example, $(a,b) = \bigcup_1^\infty [a+n^{-1}, b-n^{-1}] \in \mathcal{M}(\mathcal{E}_2)$. Verification of the other cases is left to the reader (Exercise 2). ∎

Let $\{X_\alpha\}_{\alpha\in A}$ be an indexed collection of nonempty sets, $X = \prod_{\alpha\in A} X_\alpha$, and $\pi_\alpha : X \to X_\alpha$ the coordinate maps. If $\mathcal{M}_\alpha$ is a σ-algebra on X_α for each α, the **product σ-algebra** on X is the σ-algebra generated by

$$\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha,\ \alpha \in A\}.$$

We denote this σ-algebra by $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha$. (If $A = \{1,\dots,n\}$ we also write $\bigotimes_1^n \mathcal{M}_j$ or $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$.) The significance of this definition will become clearer in §2.1; for the moment we give an alternative, and perhaps more intuitive, characterization of product σ-algebras in the case of countably many factors.

1.3 Proposition. *If A is countable, then $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha$ is the σ-algebra generated by $\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\}$.*

Proof. If $E_\alpha \in \mathcal{M}_\alpha$, then $\pi_\alpha^{-1}(E_\alpha) = \prod_{\beta\in A} E_\beta$ where $E_\beta = X$ for $\beta \neq \alpha$; on the other hand, $\prod_{\alpha\in A} E_\alpha = \bigcap_{\alpha\in A} \pi_\alpha^{-1}(E_\alpha)$. The result therefore follows from Lemma 1.1. ∎

1.4 Proposition. *Suppose that $\mathcal{M}_\alpha$ is generated by $\mathcal{E}_\alpha$, $\alpha \in A$. Then $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha$ is generated by $\mathcal{F}_1 = \{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{E}_\alpha,\ \alpha \in A\}$. If A is countable and $X_\alpha \in \mathcal{E}_\alpha$ for all α, $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha$ is generated by $\mathcal{F}_2 = \{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\}$.*

Proof. Obviously $\mathcal{M}(\mathcal{F}_1) \subset \bigotimes_{\alpha\in A} \mathcal{M}_\alpha$. On the other hand, for each α, the collection $\{E \subset X_\alpha : \pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F}_1)\}$ is easily seen to be a σ-algebra on X_α that contains $\mathcal{E}_\alpha$ and hence $\mathcal{M}_\alpha$. In other words, $\pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F}_1)$ for all $E \in \mathcal{M}_\alpha$, $\alpha \in A$, and hence $\bigotimes_{\alpha\in A} \mathcal{M}_\alpha \subset \mathcal{M}(\mathcal{F}_1)$. The second assertion follows from the first as in the proof of Proposition 1.3. ∎

1.5 Proposition. *Let $X_1, \dots, X_n$ be metric spaces and let $X = \prod_1^n X_j$, equipped with the product metric. Then $\bigotimes_1^n \mathcal{B}_{X_j} \subset \mathcal{B}_X$. If the X_j's are separable, then $\bigotimes_1^n \mathcal{B}_{X_j} = \mathcal{B}_X$.*

Proof. By Proposition 1.4, $\bigotimes_1^n \mathcal{B}_{X_j}$ is generated by the sets $\pi_j^{-1}(U_j)$, $1 \leq j \leq n$, where U_j is open in X_j. Since these sets are open in X, Lemma 1.1 implies that $\bigotimes_1^n \mathcal{B}_{X_j} \subset \mathcal{B}_X$. Suppose now that C_j is a countable dense set in X_j, and let $\mathcal{E}_j$ be the collection of balls in X_j with rational radius and center in C_j. Then every open set in X_j is a union of members of $\mathcal{E}_j$ — in fact, a countable union since $\mathcal{E}_j$ itself is countable. Moreover, the set of points in X whose jth coordinate is in C_j for all j is a countable dense subset of X, and the balls of radius r in X are merely products of balls of radius r in the X_j's. It follows that $\mathcal{B}_{X_j}$ is generated by $\mathcal{E}_j$ and $\mathcal{B}_X$ is generated by $\{\prod_1^n E_j : E_j \in \mathcal{E}_j\}$. Therefore $\mathcal{B}_X = \bigotimes_1^n \mathcal{B}_{X_j}$ by Proposition 1.4. ∎

1.6 Corollary. $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_1^n \mathcal{B}_{\mathbb{R}}$.

We conclude this section with a technical result that will be needed later. We define an **elementary family** to be a collection $\mathcal{E}$ of subsets of X such that

- $\varnothing \in \mathcal{E}$,
- if $E, F \in \mathcal{E}$ then $E \cap F \in \mathcal{E}$,
- if $E \in \mathcal{E}$ then E^c is a finite disjoint union of members of $\mathcal{E}$.

1.7 Proposition. *If $\mathcal{E}$ is an elementary family, the collection $\mathcal{A}$ of finite disjoint unions of members of $\mathcal{E}$ is an algebra.*

Proof. If $A, B \in \mathcal{E}$ and $B^c = \bigcup_1^J C_j$ ($C_j \in \mathcal{E}$, disjoint), then $A \setminus B = \bigcup_1^J (A \cap C_j)$ and $A \cup B = (A \setminus B) \cup B$, where these unions are disjoint, so $A \setminus B \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$. It now follows by induction that if $A_1 \ldots, A_n \in \mathcal{E}$, then $\bigcup_1^n A_j \in \mathcal{A}$; indeed, by inductive hypothesis we may assume that $A_1, \ldots, A_{n-1}$ are disjoint, and then $\bigcup_1^n A_j = A_n \cup \bigcup_1^{n-1}(A_j \setminus A_n)$, which is a disjoint union. To see that $\mathcal{A}$ is closed under complements, suppose $A_1, \ldots A_n \in \mathcal{E}$ and $A_m^c = \bigcup_{j=1}^{J_m} B_m^j$ with $B_m^1, \ldots, B_m^{J_m}$ disjoint members of $\mathcal{E}$. Then

$$\left(\bigcup_{m=1}^n A_m\right)^c = \bigcap_{m=1}^n \left(\bigcup_{j=1}^{J_m} B_m^j\right) = \bigcup \{B_1^{j_1} \cap \cdots \cap B_n^{j_n} : 1 \le j_m \le J_m, 1 \le m \le n\},$$

which is in $\mathcal{A}$. ∎

Exercises

1. A family of sets $\mathcal{R} \subset \mathcal{P}(X)$ is called a **ring** if it is closed under finite unions and differences (i.e., if $E_1, \ldots, E_n \in \mathcal{R}$, then $\bigcup_1^n E_j \in \mathcal{R}$, and if $E, F \in \mathcal{R}$, then $E \setminus F \in \mathcal{R}$). A ring that is closed under countable unions is called a **σ-ring**.
   a. Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.
   b. If $\mathcal{R}$ is a ring (resp. σ-ring), then $\mathcal{R}$ is an algebra (resp. σ-algebra) iff $X \in \mathcal{R}$.
   c. If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.
   d. If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

2. Complete the proof of Proposition 1.2.

3. Let $\mathcal{M}$ be an infinite σ-algebra.
   a. $\mathcal{M}$ contains an infinite sequence of disjoint sets.
   b. card($\mathcal{M}$) $\ge \mathfrak{c}$.

4. An algebra $\mathcal{A}$ is a σ-algebra iff $\mathcal{A}$ is closed under countable increasing unions (i.e., if $\{E_j\}_1^\infty \subset \mathcal{A}$ and $E_1 \subset E_2 \subset \cdots$, then $\bigcup_1^\infty E_j \in \mathcal{A}$).

5. If $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$. (Hint: Show that the latter object is a σ-algebra.)

1.3 MEASURES

Let X be a set equipped with a σ-algebra $\mathcal{M}$. A **measure** on $\mathcal{M}$ (or on $(X, \mathcal{M})$, or simply on X if $\mathcal{M}$ is understood) is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

i. $\mu(\varnothing) = 0$,

ii. if $\{E_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{M}$, then $\mu(\bigcup_1^\infty E_j) = \sum_1^\infty \mu(E_j)$.

Property (ii) is called **countable additivity**. It implies **finite additivity**:

ii′. if $E_1, \ldots E_n$ are disjoint sets in $\mathcal{M}$, then $\mu(\bigcup_1^n E_j) = \sum_1^n \mu(E_j)$,

because one can take $E_j = \varnothing$ for $j > n$. A function μ that satisfies (i) and (ii′) but not necessarily (ii) is called a **finitely additive measure**.

If X is a set and $\mathcal{M} \subset \mathcal{P}(X)$ is a σ-algebra, $(X, \mathcal{M})$ is called a **measurable space** and the sets in $\mathcal{M}$ are called **measurable sets**. If μ is a measure on $(X, \mathcal{M})$, then $(X, \mathcal{M}, \mu)$ is called a **measure space**.

Let $(X, \mathcal{M}, \mu)$ be a measure space. Here is some standard terminology concerning the "size" of μ. If $\mu(X) < \infty$ (which implies that $\mu(E) < \infty$ for all $E \in \mathcal{M}$ since $\mu(X) = \mu(E) + \mu(E^c)$), μ is called **finite**. If $X = \bigcup_1^\infty E_j$ where $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$ for all j, μ is called **σ-finite**. More generally, if $E = \bigcup_1^\infty E_j$ where $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$ for all j, the set E is said to be **σ-finite** for μ. (It would be correct but more cumbersome to say that E is of σ-finite measure.) If for each $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$, μ is called **semifinite**.

Every σ-finite measure is semifinite (Exercise 13), but not conversely. Most measures that arise in parctice are σ-finite, which is fortunate since non-σ-finite measures tend to exhibit pathological behavior. The properties of non-σ-finite measures will be explored from time to time in the exercises.

Let us examine a few examples of measures. These examples are of a rather trivial nature, although the first one is of practical importance. The construction of more interesting examples is a task to which we shall turn in the next two sections.

- Let X be any nonempty set, $\mathcal{M} = \mathcal{P}(X)$, and f any function from X to $[0, \infty]$. Then f determines a measure μ on $\mathcal{M}$ by the formula $\mu(E) = \sum_{x \in E} f(x)$. (For the definition of such possibly uncountable sums, see §0.5.) The reader may verify that μ is semifinite iff $f(x) < \infty$ for every $x \in X$, and μ is σ-finite iff μ is semifinite and $\{x : f(x) > 0\}$ is countable. Two special cases are of particular significance: If $f(x) = 1$ for all x, μ is called **counting measure**; and if, for some $x_0 \in X$, f is defined by $f(x_0) = 1$ and $f(x) = 0$ for $x \ne x_0$, μ is called the **point mass** or **Dirac measure** at x_0. (The same names are also applied to the restrictions of these measures to smaller σ-algebras on X.)

- Let X be an uncountable set, and let $\mathcal{M}$ be the σ-algebra of countable or co-countable sets. The function μ on $\mathcal{M}$ defined by $\mu(E) = 0$ if E is countable and $\mu(E) = 1$ if E is co-countable is easily seen to be a measure.

- Let X be an infinite set and $\mathcal{M} = \mathcal{P}(X)$. Define $\mu(E) = 0$ if E is finite, $\mu(E) = \infty$ if E is infinite. Then μ is a finitely additive measure but not a measure.

The basic properties of measures are summarized in the following theorem.

1.8 Theorem. *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

a. **(Monotonicity)** *If $E, F \in \mathcal{M}$ and $E \subset F$, then $\mu(E) \le \mu(F)$.*

b. **(Subadditivity)** *If $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu(\bigcup_1^\infty E_j) \le \sum_1^\infty \mu(E_j)$.*

c. **(Continuity from below)** *If* $\{E_j\}_1^\infty \subset \mathcal{M}$ *and* $E_1 \subset E_2 \subset \cdots$, *then* $\mu(\bigcup_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.

d. **(Continuity from above)** *If* $\{E_j\}_1^\infty \subset \mathcal{M}$, $E_1 \supset E_2 \supset \cdots$, *and* $\mu(E_1) < \infty$, *then* $\mu(\bigcap_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.

Proof. (a) If $E \subset F$, then $\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E)$.

(b) Let $F_1 = E_1$ and $F_k = E_k \setminus (\bigcup_1^{k-1} E_j)$ for $k > 1$. Then the F_k's are disjoint and $\bigcup_1^n F_j = \bigcup_1^n E_j$ for all n. Therefore, by (a),

$$\mu\Big(\bigcup_1^\infty E_j\Big) = \mu\Big(\bigcup_1^\infty F_j\Big) = \sum_1^\infty \mu(F_j) \leq \sum_1^\infty \mu(E_j).$$

(c) Setting $E_0 = \varnothing$, we have

$$\mu\Big(\bigcup_1^\infty E_j\Big) = \sum_1^\infty \mu(E_j \setminus E_{j-1}) = \lim_{n\to\infty} \sum_1^n \mu(E_j \setminus E_{j-1}) = \lim_{n\to\infty} \mu(E_n).$$

(d) Let $F_j = E_1 \setminus E_j$; then $F_1 \subset F_2 \subset \cdots$, $\mu(E_1) = \mu(F_j) + \mu(E_j)$, and $\bigcup_1^\infty F_j = E_1 \setminus (\bigcap_1^\infty E_j)$. By (c), then,

$$\mu(E_1) = \mu\Big(\bigcap_1^\infty E_j\Big) + \lim_{j\to\infty} \mu(F_j) = \mu\Big(\bigcap_1^\infty E_j\Big) + \lim_{j\to\infty} [\mu(E_1) - \mu(E_j)].$$

Since $\mu(E_1) < \infty$, we may subtract it from both sides to yield the desired result. ∎

We remark that the condition $\mu(E_1) < \infty$ in part (d) could be replaced by $\mu(E_n) < \infty$ for some $n > 1$, as the first $n - 1$ E_j's can be discarded from the sequence without affecting the intersection. However, some finiteness assumption is necessary, as it can happen that $\mu(E_j) = \infty$ for all j but $\mu(\bigcap_1^\infty E_j) < \infty$. (For example, let μ be counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and let $E_j = \{n : n \geq j\}$; then $\bigcap_1^\infty E_j = \varnothing$.)

If $(X, \mathcal{M}, \mu)$ is a measure space, a set $E \in \mathcal{M}$ such that $\mu(E) = 0$ is called a **null** set. By subadditivity, any countable union of null sets is a null set, a fact which we shall use frequently. If a statement about points $x \in X$ is true except for x in some null set, we say that it is true **almost everywhere** (abbreviated **a.e.**), or for **almost every** x. (If more precision is needed, we shall speak of a **μ-null** set, or **μ-almost everywhere**).

If $\mu(E) = 0$ and $F \subset E$, then $\mu(F) = 0$ by monotonicity provided that $F \in \mathcal{M}$, but in general it need not be true that $F \in \mathcal{M}$. A measure whose domain includes all subsets of null sets is called **complete**. Completeness can sometimes obviate annoying technical points, and it can always be achieved by enlarging the domain of μ, as follows.

1.9 Theorem. *Suppose that* $(X, \mathcal{M}, \mu)$ *is a measure space. Let* $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ *and* $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M}$ *and* $F \subset N$ *for some* $N \in \mathcal{N}\}$. *Then* $\overline{\mathcal{M}}$ *is a σ-algebra, and there is a unique extension* $\overline{\mu}$ *of* μ *to a complete measure on* $\overline{\mathcal{M}}$.

Proof. Since $\mathcal{M}$ and $\mathcal{N}$ are closed under countable unions, so is $\overline{\mathcal{M}}$. If $E \cup F \in \overline{\mathcal{M}}$ where $E \in \mathcal{M}$ and $F \subset N \in \mathcal{N}$, we can assume that $E \cap N = \varnothing$ (otherwise, replace F and N by $F \setminus E$ and $N \setminus E$). Then $E \cup F = (E \cup N) \cap (N^c \cup F)$, so $(E \cup F)^c = (E \cup N)^c \cup (N \setminus F)$. But $(E \cup N)^c \in \mathcal{M}$ and $N \setminus F \subset N$, so that $(E \cup F)^c \in \overline{\mathcal{M}}$. Thus $\overline{\mathcal{M}}$ is a σ-algebra.

If $E \cup F \in \overline{\mathcal{M}}$ as above, we set $\overline{\mu}(E \cup F) = \mu(E)$. This is well defined, since if $E_1 \cup F_1 = E_2 \cup F_2$ where $F_j \subset N_j \in \mathcal{N}$, then $E_1 \subset E_2 \cup N_2$ and so $\mu(E_1) \leq \mu(E_2) + \mu(N_2) = \mu(E_2)$, and likewise $\mu(E_2) \leq \mu(E_1)$. It is easily verified that $\overline{\mu}$ is a complete measure on $\overline{\mathcal{M}}$, and that $\overline{\mu}$ is the only measure on $\overline{\mathcal{M}}$ that extends μ; details are left to the reader (Exercise 6). ∎

The measure $\overline{\mu}$ in Theorem 1.9 is called the **completion** of μ, and $\overline{\mathcal{M}}$ is called the **completion** of $\mathcal{M}$ with respect to μ.

Exercises

6. Complete the proof of Theorem 1.9.

7. If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$ and $a_1, \ldots, a_n \in [0, \infty)$, then $\sum_1^n a_j\mu_j$ is a measure on $(X, \mathcal{M})$.

8. If $(X, \mathcal{M}, \mu)$ is a measure space and $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu(\liminf E_j) \leq \liminf \mu(E_j)$. Also, $\mu(\limsup E_j) \geq \limsup \mu(E_j)$ provided that $\mu(\bigcup_1^\infty E_j) < \infty$.

9. If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

10. Given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{M}$. Then μ_E is a measure.

11. A finitely additive measure μ is a measure iff it is continuous from below as in Theorem 1.8c. If $\mu(X) < \infty$, μ is a measure iff it is continuous from above as in Theorem 1.8d.

12. Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

a. If $E, F \in \mathcal{M}$ and $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$.

b. Say that $E \sim F$ if $\mu(E \Delta F) = 0$; then $\sim$ is an equivalence relation on $\mathcal{M}$.

c. For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E \Delta F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$, and hence ρ defines a metric on the space $\mathcal{M}/\sim$ of equivalence classes.

13. Every σ-finite measure is semifinite.

14. If μ is a semifinite measure and $\mu(E) = \infty$, for any $C > 0$ there exists $F \subset E$ with $C < \mu(F) < \infty$.

15. Given a measure μ on $(X, \mathcal{M})$, define μ_0 on $\mathcal{M}$ by $\mu_0(E) = \sup\{\mu(F) : F \subset E$ and $\mu(F) < \infty\}$.

a. μ_0 is a semifinite measure. It is called the **semifinite part** of μ.

b. If μ is semifinite, then $\mu = \mu_0$. (Use Exercise 14.)

c. There is a measure ν on $\mathcal{M}$ (in general, not unique) which assumes only the values 0 and ∞ such that $\mu = \mu_0 + \nu$.

16. Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \subset X$ is called **locally measurable** if $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ such that $\mu(A) < \infty$. Let $\widetilde{\mathcal{M}}$ be the collection of all locally measurable sets. Clearly $\mathcal{M} \subset \widetilde{\mathcal{M}}$; if $\mathcal{M} = \widetilde{\mathcal{M}}$, then μ is called **saturated**.

a. If μ is σ-finite, then μ is saturated.

b. $\widetilde{\mathcal{M}}$ is a σ-algebra.

c. Define $\widetilde{\mu}$ on $\widetilde{\mathcal{M}}$ by $\widetilde{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\widetilde{\mu}(E) = \infty$ otherwise. Then $\widetilde{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$, called the **saturation** of μ.

d. If μ is complete, so is $\widetilde{\mu}$.

e. Suppose that μ is semifinite. For $E \in \widetilde{\mathcal{M}}$, define $\underline{\mu}(E) = \sup\{\mu(A) : A \in \mathcal{M} \text{ and } A \subset E\}$. Then $\underline{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$ that extends μ.

f. Let X_1, X_2 be disjoint uncountable sets, $X = X_1 \cup X_2$, and $\mathcal{M}$ the σ-algebra of countable or co-countable sets in X. Let μ_0 be counting measure on $\mathcal{P}(X_1)$, and define μ on $\mathcal{M}$ by $\mu(E) = \mu_0(E \cap X_1)$. Then μ is a measure on $\mathcal{M}$, $\widetilde{\mathcal{M}} = \mathcal{P}(X)$, and in the notation of parts (c) and (e), $\widetilde{\mu} \neq \underline{\mu}$.

1.4 OUTER MEASURES

In this section we develop the tools we shall use to construct measures. To motivate the ideas, it may be useful to recall the procedure used in calculus to define the area of a bounded region E in the plane $\mathbb{R}^2$. One draws a grid of rectangles in the plane and approximates the area of E from below by the sum of the areas of the rectangles in the grid that are subsets of E, and from above by the sum of the areas of the rectangles in the grid that intersect E. The limits of these approximations as the grid is taken finer and finer give the "inner area" and "outer area" of E, and if they are equal, their common value is the "area" of E. (We shall discuss these matters in more detail in §2.6.) The key idea here is that of outer area, since if R is a large rectangle containing E, the inner area of E is just the area of R minus the outer area of $R \setminus E$.

The abstract generalization of the notion of outer area is as follows. An **outer measure** on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies

- $\mu^*(\varnothing) = 0$,
- $\mu^*(A) \le \mu^*(B)$ if $A \subset B$,
- $\mu^*(\bigcup_1^\infty A_j) \le \sum_1^\infty \mu^*(A_j)$.

The most common way to obtain outer measures is to start with a family $\mathcal{E}$ of "elementary sets" on which a notion of measure is defined (such as rectangles in the plane) and then to approximate arbitrary sets "from the outside" by countable unions of members of $\mathcal{E}$. The precise construction is as follows.

1.10 Proposition. *Let $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\varnothing \in \mathcal{E}$, $X \in \mathcal{E}$, and $\rho(\varnothing) = 0$. For any $A \subset X$, define*

$$\mu^*(A) = \inf\Big\{\sum_1^\infty \rho(E_j) : E_j \in \mathcal{E} \text{ and } A \subset \bigcup_1^\infty E_j\Big\}.$$

Then μ^* is an outer measure.

Proof. For any $A \subset X$ there exists $\{E_j\}_1^\infty \subset \mathcal{E}$ such that $A \subset \bigcup_1^\infty E_j$ (take $E_j = X$ for all j) so the definition of μ^* makes sense. Obviously $\mu^*(\varnothing) = 0$ (take $E_j = \varnothing$ for all j), and $\mu^*(A) \le \mu^*(B)$ for $A \subset B$ because the set over which the infimum is taken in the definition of $\mu^*(A)$ includes the corresponding set in the definition of $\mu^*(B)$. To prove the countable subadditivity, suppose $\{A_j\}_1^\infty \subset \mathcal{P}(X)$ and $\epsilon > 0$. For each j there exists $\{E_j^k\}_{k=1}^\infty \subset \mathcal{E}$ such that $A_j \subset \bigcup_{k=1}^\infty E_j^k$ and $\sum_{k=1}^\infty \rho(E_j^k) \le \mu^*(A_j) + \epsilon 2^{-j}$. But then if $A = \bigcup_1^\infty A_j$, we have $A \subset \bigcup_{j,k=1}^\infty E_j^k$ and $\sum_{j,k} \rho(E_j^k) \le \sum_j \mu^*(A_j) + \epsilon$, whence $\mu^*(A) \le \sum_j \mu^*(A_j) + \epsilon$. Since ϵ is arbitrary, we are done. ∎

The fundamental step that leads from outer measures to measures is as follows. If μ^* is an outer measure on X, a set $A \subset X$ is called **μ^*-measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \text{ for all } E \subset X.$$

Of course, the inequality $\mu^*(E) \le \mu^*(E \cap A) + \mu^*(E \cap A^c)$ holds for any A and E, so to prove that A is μ^*-measurable, it suffices to prove the reverse inequality. The latter is trivial if $\mu^*(E) = \infty$, so we see that A is μ^*-measurable iff

$$\mu^*(E) \ge \mu^*(E \cap A) + \mu^*(E \cap A^c) \text{ for all } E \subset X \text{ such that } \mu^*(E) < \infty.$$

Some motivation for the notion of μ^*-measurability can be obtained by referring to the discussion at the beginning of this section. If E is a "well-behaved" set such that $E \supset A$, the equation $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ says that the outer measure of A, $\mu^*(A)$, is equal to the "inner measure" of A, $\mu^*(E) - \mu^*(E \cap A^c)$. The leap from "well-behaved" sets containing A to arbitrary subsets of X a large one, but it is justified by the following theorem.

1.11 Carathéodory's Theorem. *If μ^* is an outer measure on X, the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure.*

Proof. First, we observe that $\mathcal{M}$ is closed under complements since the definition of μ^*-measurability of A is symmetric in A and A^c. Next, if $A, B \in \mathcal{M}$ and $E \subset X$,

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c)\\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c).\end{aligned}$$

But $(A \cup B) = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$, so by subadditivity,

$$\mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) \ge \mu^*(E \cap (A \cup B)),$$

and hence

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).$$

It follows that $A \cup B \in \mathcal{M}$, so $\mathcal{M}$ is an algebra. Moreover, if $A, B \in \mathcal{M}$ and $A \cap B = \varnothing$,

$$\mu^*(A \cup B) = \mu^*((A \cup B) \cap A) + \mu^*((A \cup B) \cap A^c) = \mu^*(A) + \mu^*(B),$$

so μ^* is finitely additive on $\mathcal{M}$.

To show that $\mathcal{M}$ is a σ-algebra, it will suffice to show that $\mathcal{M}$ is closed under countable disjoint unions. If $\{A_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{M}$, let $B_n = \bigcup_1^n A_j$ and $B = \bigcup_1^\infty A_j$. Then for any $E \subset X$,

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c)\\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}),\end{aligned}$$

so a simple induction shows that $\mu^*(E \cap B_n) = \sum_1^n \mu^*(E \cap A_j)$. Therefore,

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \geq \sum_1^n \mu^*(E \cap A_j) + \mu^*(E \cap B^c),$$

and letting $n \to \infty$ we obtain

$$\begin{aligned}\mu^*(E) &\geq \sum_1^\infty \mu^*(E \cap A_j) + \mu^*(E \cap B^c) \geq \mu^*\Big(\bigcup_1^\infty (E \cap A_j)\Big) + \mu^*(E \cap B^c)\\ &= \mu^*(E \cap B) + \mu^*(E \cap B^c) \geq \mu^*(E).\end{aligned}$$

All the inequalities in this last calculation are thus equalities. It follows that $B \in \mathcal{M}$ and — taking $E = B$ — that $\mu^*(B) = \sum_1^\infty \mu^*(A_j)$, so μ^* is countably additive on $\mathcal{M}$. Finally, if $\mu^*(A) = 0$, for any $E \subset X$ we have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*(E \cap A^c) \leq \mu^*(E),$$

so that $A \in \mathcal{M}$. Therefore $\mu^*|\mathcal{M}$ is a complete measure. ∎

Our first applications of Carathéodory's theorem will be in the context of extending measures from algebras to σ-algebras. More precisely, if $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, a function $\mu_0 : \mathcal{A} \to [0, \infty]$ will be called a **premeasure** if

- $\mu_0(\varnothing) = 0$,
- if $\{A_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{A}$ such that $\bigcup_1^\infty A_j \in \mathcal{A}$, then $\mu_0(\bigcup_1^\infty A_j) = \sum_1^\infty \mu_0(A_j)$.

In particular, a premeasure is finitely additive since one can take $A_j = \varnothing$ for j large. The notions of finite and σ-finite premeasures are defined just as for measures. If μ_0 is a premeasure on $\mathcal{A} \subset \mathcal{P}(X)$, it induces an outer measure on X in accordance with Proposition 1.10, namely,

$$\mu^*(E) = \inf\Big\{\sum_1^\infty \mu_0(A_j) : A_j \in \mathcal{A},\ E \subset \bigcup_1^\infty A_j\Big\}. \tag{1.12}$$

1.13 Proposition. *If μ_0 is a premeasure on $\mathcal{A}$ and μ^* is defined by (1.12), then*

a. $\mu^*|\mathcal{A} = \mu_0$;

b. every set in $\mathcal{A}$ is μ^* measurable.

Proof. (a) Suppose $E \in \mathcal{A}$. If $E \subset \bigcup_1^\infty A_j$ with $A_j \in \mathcal{A}$, let $B_n = E \cap (A_n \setminus \bigcup_1^{n-1} A_j)$. Then the B_n's are disjoint members of $\mathcal{A}$ whose union is E, so $\mu_0(E) = \sum_1^\infty \mu_0(B_j) \leq \sum_1^\infty \mu_0(A_j)$. It follows that $\mu_0(E) \leq \mu^*(E)$, and the reverse inequality is obvious since $E \subset \bigcup_1^\infty A_j$ where $A_1 = E$ and $A_j = \varnothing$ for $j > 1$.

(b) If $A \in \mathcal{A}$, $E \subset X$, and $\epsilon > 0$, there is a sequence $\{B_j\}_1^\infty \subset \mathcal{A}$ with $E \subset \bigcup_1^\infty B_j$ and $\sum_1^\infty \mu_0(B_j) \leq \mu^*(E) + \epsilon$. Since μ_0 is additive on $\mathcal{A}$,

$$\mu^*(E) + \epsilon \geq \sum_1^\infty \mu_0(B_j \cap A) + \sum_1^\infty \mu_0(B_j \cap A^c) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Since ϵ is arbitrary, A is μ^*-measurable. ∎

1.14 Theorem. *Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, μ_0 a premeasure on $\mathcal{A}$, and $\mathcal{M}$ the σ-algebra generated by $\mathcal{A}$. There exists a measure μ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0 — namely, $\mu = \mu^*|\mathcal{M}$ where μ^* is given by (1.12). If ν is another measure on $\mathcal{M}$ that extends μ_0, then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$, with equality when $\mu(E) < \infty$. If μ_0 is σ-finite, then μ is the unique extension of μ_0 to a measure on $\mathcal{M}$.*

Proof. The first assertion follows from Carathéodory's theorem and Proposition 1.13 since the σ-algebra of μ^*-measurable sets includes $\mathcal{A}$ and hence $\mathcal{M}$. As for the second assertion, if $E \in \mathcal{M}$ and $E \subset \bigcup_1^\infty A_j$ where $A_j \in \mathcal{A}$, then $\nu(E) \leq \sum_1^\infty \nu(A_j) = \sum_1^\infty \mu_0(A_j)$, whence $\nu(E) \leq \mu(E)$. Also, if we set $A = \bigcup_1^\infty A_j$, we have

$$\nu(A) = \lim_{n\to\infty} \nu\Big(\bigcup_1^n A_j\Big) = \lim_{n\to\infty} \mu\Big(\bigcup_1^n A_j\Big) = \mu(A).$$

If $\mu(E) < \infty$, we can choose the A_j's so that $\mu(A) < \mu(E) + \epsilon$, hence $\mu(A \setminus E) < \epsilon$, and

$$\mu(E) \leq \mu(A) = \nu(A) = \nu(E) + \nu(A \setminus E) \leq \nu(E) + \mu(A \setminus E) \leq \nu(E) + \epsilon.$$

Since ϵ is arbitrary, $\mu(E) = \nu(E)$. Finally, suppose $X = \bigcup_1^\infty A_j$ with $\mu_0(A_j) < \infty$, where we can assume that the A_j's are disjoint. Then for any $E \in \mathcal{M}$,

$$\mu(E) = \sum_1^\infty \mu(E \cap A_j) = \sum_1^\infty \nu(E \cap A_j) = \nu(E),$$

so $\nu = \mu$. ∎

44. **(Lusin's Theorem)** If $f : [a, b] \to \mathbb{C}$ is Lebesgue measurable and $\epsilon > 0$, there is a compact set $E \subset [a, b]$ such that $\mu(E^c) < \epsilon$ and $f|E$ is continuous. (Use Egoroff's theorem and Theorem 2.26.)

2.5 PRODUCT MEASURES

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. We have already discussed the product σ-algebra $\mathcal{M} \otimes \mathcal{N}$ on $X \times Y$; we now construct a measure on $\mathcal{M} \otimes \mathcal{N}$ that is, in an obvious sense, the product of μ and ν.

To begin with, we define a (measurable) **rectangle** to be a set of the form $A \times B$ where $A \in \mathcal{M}$ and $B \in \mathcal{N}$. Clearly

$$(A \times B) \cap (E \times F) = (A \cap E) \times (B \cap F), \qquad (A \times B)^c = (X \times B^c) \cup (A^c \times B).$$

Therefore, by Proposition 1.7, the collection $\mathcal{A}$ of finite disjoint unions of rectangles is an algebra, and of course the σ-algebra it generates is $\mathcal{M} \otimes \mathcal{N}$.

Suppose $A \times B$ is a rectangle that is a (finite or countable) disjoint union of rectangles $A_j \times B_j$. Then for $x \in X$ and $y \in Y$,

$$\chi_A(x)\chi_B(y) = \chi_{A \times B}(x, y) = \sum \chi_{A_j \times B_j}(x, y) = \sum \chi_{A_j}(x)\chi_{B_j}(y).$$

If we integrate with respect to x and use Theorem 2.15, we obtain

$$\mu(A)\chi_B(y) = \int \chi_A(x)\chi_B(y)\, d\mu(x) = \sum \int \chi_{A_j}(x)\chi_{B_j}(y)\, d\mu(x) = \sum \mu(A_j)\chi_{B_j}(y).$$

In the same way, integration in y then yields

$$\mu(A)\nu(B) = \sum \mu(A_j)\nu(B_j).$$

It follows that if $E \in \mathcal{A}$ is the disjoint union of rectangles $A_1 \times B_1, \ldots, A_n \times B_n$, and we set

$$\pi(E) = \sum_1^n \mu(A_j)\nu(E_j)$$

(with the usual convention that $0 \cdot \infty = 0$), then π is well defined on $\mathcal{A}$ (since any two representations of E as a finite disjoint union of rectangles have a common refinement), and π is a premeasure on $\mathcal{A}$. According to Theorem 1.14, therefore, π generates an outer measure on $X \times Y$ whose restriction to $\mathcal{M} \times \mathcal{N}$ is a measure that extends π. We call this measure the **product** of μ and ν and denote it by $\mu \times \nu$. Moreover, if μ and ν are σ-finite — say, $X = \bigcup_1^\infty A_j$ and $Y = \bigcup_1^\infty B_k$ with $\mu(A_j) < \infty$ and $\nu(B_k) < \infty$ — then $X \times Y = \bigcup_{j,k} A_j \times B_k$, and $\mu \times \nu(A_j \times B_k) < \infty$, so $\mu \times \nu$ is also σ-finite. In this case, by Theorem 1.14, $\mu \times \nu$ is the unique measure on $\mathcal{M} \otimes \mathcal{N}$ such that $\mu \times \nu(A \times B) = \mu(A)\nu(B)$ for all rectangles $A \times B$.

The same construction works for any finite number of factors. That is, suppose $(X_j, \mathcal{M}_j, \mu_j)$ are measure spaces for $j = 1, \ldots, n$. If we define a rectangle to be a set of the form $A_1 \times \cdots \times A_n$ with $A_j \in \mathcal{M}_j$, then the collection $\mathcal{A}$ of finite disjoint unions of rectangles is an algebra, and the same procedure as above produces a measure $\mu_1 \times \cdots \times \mu_n$ on $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ such that

$$\mu_1 \times \cdots \times \mu_n(A_1 \times \cdots \times A_n) = \prod_1^n \mu_j(A_j).$$

Moreover, if the μ_j's are σ-finite so that the extension from $\mathcal{A}$ to $\bigotimes_1^n \mathcal{M}_j$ is uniquely determined, the obvious associativity properties hold. For example, if we identify $X_1 \times X_2 \times X_3$ with $(X_1 \times X_2) \times X_3$, we have $\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \mathcal{M}_3 = (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$ (the former being generated by sets of the form $A_1 \times A_2 \times A_3$ with $A_j \in \mathcal{M}_j$, and the latter by sets of the form $B \times A_3$ with $B \in \mathcal{M}_1 \otimes \mathcal{M}_2$ and $A_3 \in \mathcal{M}_3$), and $\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$ (since they agree on sets of the form $A_1 \times A_2 \times A_3$, and hence in general by uniqueness). Details are left to the reader (Exercise 45). All of our results below have obvious extensions to products with n factors, but we shall stick to the case $n = 2$ for simplicity.

We return to the case of two measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$. If $E \subset X \times Y$, for $x \in X$ and $y \in Y$ we define the **x-section** E_x and the **y-section** E^y of E by

$$E_x = \{y \in Y : (x, y) \in E\}, \qquad E^y = \{x \in X : (x, y) \in E\}.$$

Also, if f is a function on $X \times Y$ we define the **x-section** f_x and the **y-section** f^y of f by

$$f_x(y) = f^y(x) = f(x, y).$$

Thus, for example, $(\chi_E)_x = \chi_{E_x}$ and $(\chi_E)^y = \chi_{E^y}$.

2.34 Proposition.

a. *If $E \in \mathcal{M} \times \mathcal{N}$, then $E_x \in \mathcal{N}$ for all $x \in X$ and $E^y \in \mathcal{M}$ for all $y \in Y$.*

b. *If f is $\mathcal{M} \otimes \mathcal{N}$-measurable, then f_x is $\mathcal{N}$-measurable for all $x \in X$ and f^y is $\mathcal{M}$-measurable for all $y \in Y$.*

Proof. Let $\mathcal{R}$ be the collection of all subsets E of $X \times Y$ such that $E_x \in \mathcal{N}$ for all x and $E^y \in \mathcal{M}$ for all y. Then $\mathcal{R}$ obviously contains all rectangles (e.g., $(A \times B)_x = B$ if $x \in A$, $= \varnothing$ otherwise). Since $(\bigcup_1^\infty E_j)_x = \bigcup_1^\infty (E_j)_x$ and $(E^c)_x = (E_x)^c$, and likewise for y-sections, $\mathcal{R}$ is a σ-algebra. Therefore $\mathcal{R} \supset \mathcal{M} \otimes \mathcal{N}$, which proves (a). (b) follows from (a) because $(f_x)^{-1}(B) = (f^{-1}(B))_x$ and $(f^y)^{-1}(B) = (f^{-1}(B))^y$. ∎

Before proceeding further we need a technical lemma. We define a **monotone class** on a space X to be a subset $\mathcal{C}$ of $\mathcal{P}(X)$ that is closed under countable increasing unions and countable decreasing intersections (that is, if $E_j \in \mathcal{C}$ and $E_1 \subset E_2 \subset \cdots$, then $\bigcup E_j \in \mathcal{C}$, and likewise for intersections). Clearly every σ-algebra is a monotone class. Also, the intersection of any family of monotone classes is a monotone class,

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so for any $\mathcal{E} \subset \mathcal{P}(X)$ there is a unique smallest monotone class containing $\mathcal{E}$, called the monotone class **generated by** $\mathcal{E}$.

2.35 The Monotone Class Lemma. *If $\mathcal{A}$ is an algebra of subsets of X, then the monotone class $\mathcal{C}$ generated by $\mathcal{A}$ coincides with the σ-algebra $\mathcal{M}$ generated by $\mathcal{A}$.*

Proof. Since $\mathcal{M}$ is a monotone class, we have $\mathcal{C} \subset \mathcal{M}$; and if we can show that $\mathcal{C}$ is a σ-algebra, we will have $\mathcal{M} \subset \mathcal{C}$. To this end, for $E \in \mathcal{C}$ let us define

$$\mathcal{C}(E) = \{F \in \mathcal{C} : E \setminus F,\ F \setminus E, \text{ and } E \cap F \text{ are in } \mathcal{C}\}.$$

Clearly $\varnothing$ and E are in $\mathcal{C}(E)$, and $E \in \mathcal{C}(F)$ iff $F \in \mathcal{C}(E)$. Also, it is easy to check that $\mathcal{C}(E)$ is a monotone class. If $E \in \mathcal{A}$, then $F \in \mathcal{C}(E)$ for all $F \in \mathcal{A}$ because $\mathcal{A}$ is an algebra; that is, $\mathcal{A} \subset \mathcal{C}(E)$, and hence $\mathcal{C} \subset \mathcal{C}(E)$. Therefore, if $F \in \mathcal{C}$, then $F \in \mathcal{C}(E)$ for all $E \in \mathcal{A}$. But this means that $E \in \mathcal{C}(F)$ for all $E \in \mathcal{A}$, so that $\mathcal{A} \subset \mathcal{C}(F)$ and hence $\mathcal{C} \subset \mathcal{C}(F)$. Conclusion: If $E, F \in \mathcal{C}$, then $E \setminus F$ and $E \cap F$ are in $\mathcal{C}$. Since $X \in \mathcal{A} \subset \mathcal{C}$, $\mathcal{C}$ is therefore an algebra. But then if $\{E_j\}_1^\infty \subset \mathcal{C}$, we have $\bigcup_1^n E_j \in \mathcal{C}$ for all n, and since $\mathcal{C}$ is closed under countable increasing unions it follows that $\bigcup_1^\infty E_j \in \mathcal{C}$. In short, $\mathcal{C}$ is a σ-algebra, and we are done. ∎

We now come to the main results of this section, which relate integrals on $X \times Y$ to integrals on X and Y.

2.36 Theorem. *Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$, then the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable on X and Y, respectively, and*

$$\mu \times \nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y).$$

Proof. First suppose that μ and ν are finite, and let $\mathcal{C}$ be the set of all $E \in \mathcal{M} \otimes \mathcal{N}$ for which the conclusions of the theorem are true. If $E = A \times B$, then $\nu(E_x) = \chi_A(x)\nu(B)$ and $\mu(E^y) = \mu(A)\chi_B(y)$, so clearly $E \in \mathcal{C}$. By additivity it follows that finite disjoint unions of rectangles are in $\mathcal{C}$, so by Lemma 2.35 it will suffice to show that $\mathcal{C}$ is a monotone class. If $\{E_n\}$ is an increasing sequence in $\mathcal{C}$ and $E = \bigcup_1^\infty E_n$, then the functions $f_n(y) = \mu((E_n)^y)$ are measurable and increase pointwise to $f(y) = \mu(E^y)$. Hence f is measurable, and by the monotone convergence theorem,

$$\int \mu(E^y)\, d\nu(y) = \lim \int \mu((E_n)^y)\, d\nu(y) = \lim \mu \times \nu(E_n) = \mu \times \nu(E).$$

Likewise $\mu \times \nu(E) = \int \nu(E_x)\, d\mu(x)$, so $E \in \mathcal{C}$. Similarly, if $\{E_n\}$ is a decreasing sequence in $\mathcal{C}$ and $\bigcap_1^\infty E_n$, the function $y \mapsto \mu((E_1)^y)$ is in $L^1(\nu)$ because $\mu((E_1)^y) \le \mu(X) < \infty$ and $\nu(Y) < \infty$, so the dominated convergence theorem can be applied to show that $E \in \mathcal{C}$. Thus $\mathcal{C}$ is a monotone class, and the proof is complete for the case of finite measure spaces.

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Finally, if μ and ν are σ-finite, we can write $X \times Y$ as the union of an increasing sequence $\{X_j \times Y_j\}$ of rectangles of finite measure. If $E \in \mathcal{M} \otimes \mathcal{N}$, the preceding argument applies to $E \cap (X_j \times Y_j)$ for each j to give

$$\mu\times\nu(E\cap(X_j\times Y_j)) = \int \chi_{X_j}(x)\nu(E_x \cap Y_j)\, d\mu(x) = \int \chi_{Y_j}(y)\mu(E^y \cap X_j)\, d\nu(y),$$

and a final application of the monotone convergence theorem then yields the desired result. ∎

2.37 The Fubini-Tonelli Theorem. *Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces.*

a. (Tonelli) *If $f \in L^+(X \times Y)$, then the functions $g(x) = \int f_x\, d\nu$ and $h(y) = \int f^y\, d\mu$ are in $L^+(X)$ and $L^+(Y)$, respectively, and*

$$\int f\, d(\mu \times \nu) = \int \left[\int f(x,y)\, d\nu(y)\right] d\mu(x) \tag{2.38}$$
$$= \int \left[\int f(x,y)\, d\mu(x)\right] d\nu(y).$$

b. (Fubini) *If $f \in L^1(\mu \times \nu)$, then $f_x \in L^1(\nu)$ for a.e. $x \in X$, $f^y \in L^1(\mu)$ for a.e. $y \in Y$, the a.e.-defined functions $g(x) = \int f_x\, d\nu$ and $h(x) = \int f^y\, d\nu$ are in $L^1(\mu)$ and $L^1(\nu)$, respectively, and (2.38) holds.*

Proof. Tonelli's theorem reduces to Theorem 2.36 in case f is a characteristic function, and it therefore holds for nonnegative simple functions by linearity. If $f \in L^+(X \times Y)$, let $\{f_n\}$ be a sequence of simple functions that increase pointwise to f as in Theorem 2.10. The monotone convergence theorem implies, first, that the corresponding g_n and h_n increase to g and h (so that g and h are measurable), and, second that

$$\int g\, d\mu = \lim \int g_n\, d\mu = \lim \int f_n\, d(\mu \times \nu) = \int f\, d(\mu \times \nu),$$
$$\int h\, d\nu = \lim \int h_n\, d\nu = \lim \int f_n\, d(\mu \times \nu) = \int f\, d(\mu \times \nu),$$

which is (2.38). This establishes Tonelli's theorem and also shows that if $f \in L^+(X \times Y)$ and $\int f\, d(\mu \times \nu) < \infty$, then $g < \infty$ a.e. and $h < \infty$ a.e., that is, $f_x \in L^1(\nu)$ for a.e. x and $f^y \in L^1(\mu)$ for a.e. y. If $f \in L^1(\mu \times \nu)$, then, the conclusion of Fubini's theorem follows by applying these results to the positive and negative parts of the real and imaginary parts of f. ∎

A few remarks are in order:

- We shall usually omit the brackets in the iterated integrals in (2.38), thus:

$$\int \left[\int f(x,y)\, d\mu(x)\right] d\nu(y) = \iint f(x,y)\, d\mu(x)\, d\nu(y) = \iint f\, d\mu\, d\nu.$$

2.6 THE n-DIMENSIONAL LEBESGUE INTEGRAL

Lebesgue measure m^n on $\mathbb{R}^n$ is the completion of the n-fold product of Lebesgue measure on $\mathbb{R}$ with itself, that is, the completion of $m \times \cdots \times m$ on $\mathcal{B}_{\mathbb{R}} \otimes \cdots \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^n}$, or equivalently the completion of $m \times \cdots \times m$ on $\mathcal{L} \otimes \cdots \otimes \mathcal{L}$. The domain $\mathcal{L}^n$ of m^n is the class of **Lebesgue measurable** sets in $\mathbb{R}^n$; sometimes we shall also consider m^n as a measure on the smaller domain $\mathcal{B}_{\mathbb{R}^n}$. When there is no danger of confusion, we shall usually omit the superscript n and write m for m^n, and as in the case $n = 1$, we shall usually write $\int f(x)\,dx$ for $\int f\,dm$.

We begin by establishing the extensions of some of the results in §1.5 to the n-dimensional case. In what follows, if $E = \prod_1^n E_j$ is a rectangle in $\mathbb{R}^n$, we shall refer to the sets $E_j \subset \mathbb{R}$ as the **sides** of E.

2.40 Theorem. *Suppose $E \in \mathcal{L}^n$.*

a. $m(E) = \inf\{m(U) : U \supset E,\ U \text{ open}\} = \sup\{m(K) : K \subset E,\ K \text{ compact}\}$.

b. $E = A_1 \cup N_1 = A_2 \setminus N_2$ where A_1 is an F_σ set, A_2 is a G_δ set, and $m(N_1) = m(N_2) = 0$.

c. If $m(E) < \infty$, for any $\epsilon > 0$ there is a finite collection $\{R_j\}_1^N$ of disjoint rectangles whose sides are intervals such that $m(E \triangle \bigcup_1^N R_j) < \epsilon$.

Proof. By the definition of product measures, if $E \in \mathcal{L}^n$ and $\epsilon > 0$ there is a countable family $\{T_j\}$ of rectangles such that $E \subset \bigcup_1^\infty T_j$ and $\sum_1^\infty m(T_j) \le m(E) + \epsilon$. For each j, by applying Theorem 1.18 to the sides of R_j we can find a rectangle $U_j \supset F_j$ whose sides are open sets such that $m(U_j) < m(T_j) + \epsilon 2^{-j}$. If $U = \bigcup_1^\infty U_j$, then U is open and $m(U) \le \sum_1^\infty m(U_j) \le m(E) + 2\epsilon$. This proves the first equation in part (a); the second one, and part (b), then follow as in the proofs of Theorems 1.18 and 1.19. Next, if $m(E) < \infty$, then $m(U_j) < \infty$ for all j. Since the sides of U_j are countable unions of open intervals, by taking suitable finite subunions we obtain rectangles $V_j \subset U_j$ whose sides are finite unions of intervals such that $m(V_j) \ge m(U_j) - \epsilon 2^{-j}$. If N is sufficiently large, then, we have

$$m\Big(E \setminus \bigcup_1^N V_j\Big) \le m\Big(\bigcup_1^N U_j \setminus V_j\Big) + m\Big(\bigcup_{N+1}^\infty U_j\Big) < 2\epsilon$$

and

$$m\Big(\bigcup_1^N V_j \setminus E\Big) \le m\Big(\bigcup_1^\infty U_j \setminus E\Big) < \epsilon,$$

so that $m(E \triangle \bigcup_1^N V_j) < 3\epsilon$. Since $\bigcup_1^N V_j$ can be expressed as a finite disjoint union of rectangles whose sides are intervals, we have proved (c). ∎

2.41 Theorem. *If $f \in L^1(m)$ and $\epsilon > 0$, there is a simple function $\phi = \sum_1^N a_j \chi_{R_j}$, where each R_j is a product of intervals, such that $\int |f - \phi| < \epsilon$, and there is a continuous function g that vanishes outside a bounded set such that $\int |f - g| < \epsilon$.*

Theorems 1.18–1.19 = Th. 16.21 in Carothers

= Th. 18.27 in Carothers

Proof. As in the proof of Theorem 2.26, approximate f by simple functions, then use Theorem 2.40c to approximate the latter by functions ϕ of the desired form. Finally, approximate such ϕ's by continuous functions by applying an obvious generalization of the argument in the proof of Theorem 2.26. ∎

2.42 Theorem. *Lebesgue measure is translation-invariant. More precisely, for $a \in \mathbb{R}^n$ define $\tau_a : \mathbb{R}^n \to \mathbb{R}^n$ by $\tau_a(x) = x + a$.*

a. If $E \in \mathcal{L}^n$, then $\tau_a(E) \in \mathcal{L}^n$ and $m(\tau_a(E)) = m(E)$.

b. If $f : \mathbb{R}^n \to \mathbb{C}$ is Lebesgue measurable, then so is $f \circ \tau_a$. Moreover, if either $f \ge 0$ or $f \in L^1(m)$, then $\int (f \circ \tau_a)\,dm = \int f\,dm$.

Proof. Since τ_a and its inverse τ_{-a} are continuous, they preserve the class of Borel sets. The formula $m(\tau_a(E)) = m(E)$ follows easily from the one-dimensional result (Theorem 1.21) if E is a rectangle, and it then follows for general Borel sets since m is determined by its action on rectangles (the uniqueness in Theorem 1.14). In particular, the collection of Borel sets E such that $m(E) = 0$ is invariant under τ_a. Assertion (a) now follows immediately.

If f is Lebesgue measurable and B is a Borel set in $\mathbb{C}$, we have $f^{-1}(B) = E \cup N$ where E is Borel and $m(N) = 0$. But $\tau_a^{-1}(E)$ is Borel and $m(\tau_a^{-1}(N)) = 0$, so $(f \circ \tau_a)^{-1}(B) \in \mathcal{L}^n$ and f is Lebesgue measurable. The equality $\int (f \circ \tau_a)\,d\mu = \int f\,d\mu$ reduces to the equality $m(\tau_{-a}(E)) = m(E)$ when $f = \chi_E$. It is then true for simple functions by linearity, and hence for nonnegative measurable functions by the definition of the integral. Taking positive and negative parts of real and imaginary parts then yields the result for $f \in L^1(m)$. ∎

Let us now compare Lebesgue measure on $\mathbb{R}^n$ to the more naive theory of n-dimensional measure usually found in advanced calculus books. In this discussion, a **cube** in $\mathbb{R}^n$ is a Cartesian product of n closed intervals whose side lengths are all equal.

For $k \in \mathbb{Z}$, let $\mathcal{Q}_k$ be the collection of cubes whose side length is 2^{-k} and whose vertices are in the lattice $(2^{-k}\mathbb{Z})^n$. (That is, $\prod_1^n [a_j, b_j] \in \mathcal{Q}_k$ iff $2^k a_j$ and $2^k b_j$ are integers and $b_j - a_j = 2^{-k}$ for all j.) Note that any two cubes in $\mathcal{Q}_k$ have disjoint interiors, and that the cubes in $\mathcal{Q}_{k+1}$ are obtained from the cubes in $\mathcal{Q}_k$ by bisecting the sides.

If $E \subset \mathbb{R}^n$, we define the inner and outer approximations to E by the grid of cubes $\mathcal{Q}_k$ to be

$$\underline{A}(E,k) = \bigcup\{Q \in \mathcal{Q}_k : Q \subset E\}, \qquad \overline{A}(E,k) = \bigcup\{Q \in \mathcal{Q}_k : Q \cap E \neq \varnothing\}.$$

(See Figure 2.2.) The measure of $\underline{A}(E,k)$ (in either the naive geometric sense or the Lebesgue sense) is just 2^{-nk} times the number of cubes in $\mathcal{Q}_k$ that lie in $\underline{A}(E,k)$, and we denote it by $m(\underline{A}(E,k))$; likewise for $m(\overline{A}(E,k))$. Also, the sets $\underline{A}(E,k)$ increase with k while the sets $\overline{A}(E,k)$ decrease, because each cube in $\mathcal{Q}_k$ is a union of cubes in $\mathcal{Q}_{k+1}$. Hence the limits

$$\underline{\kappa}(E) = \lim_{k\to\infty} m(\underline{A}(E,k)), \qquad \overline{\kappa}(E) = \lim_{k\to\infty} m(\overline{A}(E,k))$$