Unfortunately, the posted solution to Problem 5 of HW8 is rather terse. We provide some
details below. From the given data we have the following short exact sequence of groups,

\[ 1 \to \ker(\alpha) \to G \overset{\alpha}{\to} H \to 1. \]

Here the group operation has been written multiplicatively and exactness means that at
each intermediate term, i.e., \( \ker(\alpha), G \) and \( H \), the image of the preceding map is equal to
the kernel of the following map. Clearly, \( \ker(\alpha) \) is a normal subgroup of \( G \). It is also given
that there is a group homomorphism \( \beta : H \to G \), which ‘splits’ the above sequence, i.e.,
\( \alpha \beta = \text{id}_H \). It follows that \( \beta(H) \) is a subgroup of \( G \). We can produce a map

\[ \theta : H \to \text{Aut}(\ker(\alpha)) \]

\[ h \mapsto [a \mapsto \beta(h)a\beta(h)^{-1}] \]

where \( a \in \ker(\alpha) \). Since \( \beta(H) \) is a subgroup of \( G \) it acts on \( G \) by conjugation and \( \theta \) is the
restriction of that action to \( \ker(\alpha) \). In order to be able to restrict the action, we need to
check that for all \( a \in \ker(\alpha) \), one has \( \beta(h)a\beta(h)^{-1} \in \ker(\alpha) \). This is, indeed, true:

\[ \alpha(\beta(h)a\beta(h)^{-1}) = \alpha\beta(h)\alpha(a)(\alpha\beta(h))^{-1} = 1 \quad [\text{since } \alpha(a) = 1] \]

Therefore, the map \( \theta \) is actually a group homomorphism (cf. the EXTRA PROBLEM of
HW7). Now consider the group homomorphism

\[ \phi : G \to \ker(\alpha) \rtimes_\theta H \]

\[ g \mapsto (gh^{-1}, \alpha(g)), \]

where \( h = \beta\alpha(g) \). We need to check that \( gh^{-1} \in \ker(\alpha) \) and that \( \phi \) respects group operations.
Indeed, one finds

\[ \alpha(gh^{-1}) = \alpha(g)(\alpha(\beta\alpha(g)))^{-1} = \alpha(g)\alpha(g)^{-1} = 1 \quad [\text{since } \alpha\beta = \text{id}_H] \]

\[ \phi(g_1g_2) = \alpha(g_1)\alpha(g_2) \]
\[ = \alpha(g_1)\alpha(g_2)^{-1} \]
\[ = \alpha(g_1)\alpha(g_2)^{-1} \]
\[ = (g_1\alpha(g_1)^{-1})(g_2\alpha(g_2)^{-1}, \alpha(g_2)) \]
\[ = (g_1\alpha(g_1)^{-1})(g_2\alpha(g_2)^{-1}, \alpha(g_2)) \]
\[ = (g_1\alpha(g_1)^{-1})(g_2\alpha(g_2)^{-1}, \alpha(g_2)) \]
\[ = (g_1\alpha(g_1)^{-1})(g_2\alpha(g_2)^{-1}, \alpha(g_2)) \]
\[ = (g_1\alpha(g_1)^{-1})(g_2\alpha(g_2)^{-1}, \alpha(g_2)) \]

Now you may use the Short Five Lemma argument that was posted originally.