1. Let $p \in \text{Spec } R$

By problem 2 of the last hw, we have an injection

$$\left\{ \begin{array}{c}
p \in \text{Spec } R \\
p \in \text{Spec } S = \emptyset
\end{array} \right\} \hookrightarrow \left\{ \text{Spec } S - R \right\}$$

when $e$ is extension and $c$ is contraction.

Now $Rp$ is a local ring since the ideal $pRp$ is exactly the set of elements of $Rp$ that are not units; $\frac{R_p}{pRp}$ is a field.

$\therefore$ by hw 2, Question 4, $(R_p, pR_p)$ is a local ring.

2. Let $H$ be an abelian group.

Then $\text{End}_2(M)$ is a ring with addition defined as $(\varphi + \psi)(m) = \varphi(m) + \psi(m)$ and multiplication defined as composition.

Note this ring contains the identity endomorphism; $\cdot$ is unital.

Let $R$ be a ring, $\varphi: R \to \text{End}_2(M)$ a ring homomorphism.

Then $M$ is an $R$-module, where $R \times M \to M$ is defined by $(r, m) \mapsto \varphi(r)(m)$.

Conversely, let $M$ be an $R$-module. Then there is an action $R \times M \to M$ defined by $r \cdot m = r m$.

Then $\varphi: R \to \text{End}_2(M)$ can be defined by $r \mapsto \varphi(r): M \to M$.

3. Let $0 \to M \to N \to Q \to 0$ be a short exact seq. of $R$-mod.

a) By exactness, $\ker q = \text{im } f$, $\therefore N/\text{im } f \cong N/\ker f$.

Now, $N/\ker f$ is a map $\cdot: \text{im } f = \ker g$ get set to $0$.

:. by universal property of quotient construction, it factors $N \to \ker h$.

It is an isomorphism by the $1^{st}$ Iso Thm of groups, noting that the module action by $R$ is preserved.

b) Consider the SES

$$0 \to \text{ker } f \to M \to \text{im } f \to 0$$

Then apply a) to get $N/\text{ker } f \cong \text{im } f$.

c) Let $M, N$ simple $R$-mod and $f: M \to N$ an $R$-mod homomorphism.

If $f$ nonzero, $\ker f \neq M$; since ker $f$ is a submodule, must be $0$ $\therefore$ f injective

also, if $f$ nonzero, im $f$, which is a submodule of $N$ must be $N$ $\therefore$ f surjective

:. $f \in \text{End}_2(M)$, $f$ is an isomorphism $\therefore$ has an inverse.

Since $M \to N$ is in $\text{End}_2(M)$, $\text{End}_2(M)$ is a division ring.
4. Let $\mathcal{P}$ be a subcategory.

Let $P$ be projective.

Consider the SES: $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$.

Then since $P$ is projective, using $\mathcal{P}(P) \rightarrow P$ the identity, we get a lift $k: P \rightarrow N$.

\[ k: P \rightarrow N \rightarrow N \rightarrow P \rightarrow 0 \]

\[ \therefore \text{SES splits} \]

Conversely, for any SES $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$, there exists a splitting homomorphism $s: P \rightarrow N$ splitting homomorphism.

Consider the diagram:

\[ 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \]

To show $P$ is projective we have to show $3$ lift $P \rightarrow M''$.

Every module is the quotient of a free module, (viz: free module on set of elements of $P$).

\[ F\alpha \quad \exists \beta \quad \exists \gamma \]

we have SES $0 \rightarrow ker \alpha \rightarrow F\alpha \rightarrow P \rightarrow 0$.

By assumption, $P$ splits $F\alpha$.

Then define $s: P \rightarrow N$ by $s: P \rightarrow F\alpha \rightarrow N$, since $F\alpha$ exists by the universal property of free modules.

defined by mapping $X \in X$ to the pullback of their images in $N$.

5. Let $R, M \neq 0$

Then $R \cong M$, def by $(r, m) \mapsto 0$ clearly gives $M$ a $R$-mod structure.

Assume $R$ is free on a set $X = \{ x_i \}_{i \in I}$. Consider a nonzero map $j: X \rightarrow R^{\text{metrified}}$.

By universal property, this map factors:

\[ X \cong F(X) \cong R^{\text{triv}} \]

\[ \therefore \text{the diagram commutes} \]

Then $0 = 1 \cdot k(i(x)) = k(1 \cdot i(x)) = k(i(x)) \quad \forall x \in X$.

\[ k \equiv 0 \]

\[ \therefore \text{since } j \text{ is nonzero, diagram can not commute} \]