Let $F|k$ be any field extension, i.e., $k \subset F$, and let $t$ be an indeterminate variable throughout.

(1) Let $F|k$ be a simple extension, i.e., $F = k(u)$, for some $u \in F \setminus k$. If $u$ is algebraic over $k$, then show that $\{1, u, \ldots, u^{n-1}\}$ is a basis of $F$ as a vector space over $k$.

(2) If $F$ is a finite dimensional vector space over $k$, then show that $F$ is finitely generated and algebraic over $k$.

(3) Consider the element $u = \frac{t^3}{1+t}$ in the field $k(t)$. Show that $k(t)$ is a simple extension over $k(u)$. What is $[k(t) : k(u)]$?

(4) Show that $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2})$ are isomorphic as $\mathbb{Q}$-vector spaces, but not as subfields of $\mathbb{C}$.

(5) Show that $f(t) = t^5 + 2t + 2$ is an irreducible polynomial in $\mathbb{Q}[t]$. Let $u$ be a real root of $f(t)$ and let $\mathbb{Q}(u) \subset \mathbb{R}$ be the subfield generated by $\mathbb{Q}$ and $u$. Express the elements $(u^2 + 2)(u^3 + 3u)$ and $u^{-1}$ as a $\mathbb{Q}$-linear combination in terms of the basis $\{1, u, u^2, u^3, u^4\}$.

EXTRA PROBLEM: (not to be graded)
Let $F|k$ be a field extension. Set $E = \{x \in F \mid x$ is algebraic over $k\}$. Show that $E|k$ is an algebraic extension of fields. [Hint: It suffices to check that $E \subset F$ is a subfield, i.e., whenever $u, v$ belong to $E$, both $u - v$ and $uv^{-1}$ belong to $E$.]