Name (in block capitals): _______________________________________________________

Signature: ___________________________________________________________________

(1) Solve all the problems and read each one very carefully before answering it.
(2) This is a closed book, closed notes examination. The use of calculators or discussions during the exam is NOT permitted.
(3) A correct answer with incomplete arguments would NOT guarantee full credit. However, an incorrect answer may be awarded partial credit if some correct and non-trivial steps are shown.
(4) The use of pencils is NOT permitted.
(5) Indicate clearly in your solutions the logical sequence of your steps. Credit may NOT be awarded if the handwriting is illegible.
(6) The rough work will NOT be taken into consideration for grading purposes.
(7) If you want to use a non-standard fact that you know, you MUST state it clearly.

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**DO NOT WRITE ANYTHING ON THE SCORE-SHEET.**

Unless otherwise stated, do not assume rings to be unital or commutative. Do not assume ring homomorphisms to be unital (between unital rings) and modules over unital rings to be unitary.
(1) [3+7 points]

Let \( R \) be a ring. Let \( M' \subset M, N' \subset N \) be \( R \)-submodules of \( R \)-modules \( M, N \). An \( R \)-module homomorphism \( f : M \to N \) is called special if \( f(M') \subset N' \).

(a) Show that any special nonzero \( R \)-module homomorphism induces canonically an \( R \)-module homomorphism \( \overline{f} : M/M' \to N/N' \), which is nonzero unless \( M' = M \) or \( N' = N \).

(b) Now assume further \( R \) to be unital and all modules over it to be unitary. Suppose \( M' \neq M \) and \( N' \neq N \) such that \( M/M' \) and \( N/N' \) are simple as \( R \)-modules. Let \( f : M \to N \) be a nonzero special \( R \)-module homomorphism. Prove that \( f : M \to N \) is an \( R \)-module isomorphism if and only if so is the restricted homomorphism \( f : M' \to N' \). [Observe that the restriction makes sense since \( f \) is special.]
(2) [5+5 points]
A ring homomorphism $f : R \rightarrow S$ is an abelian group homomorphism such that $f(rr') = f(r)f(r')$ for all $r, r' \in R$. Let $R$ be ring with identity.
(a) For any $x \in R$ show that the map $R \rightarrow xR$ sending $r \mapsto xr$ is a ring homomorphism if $x^2 = x$ and $x$ commutes with every $r \in R$. [Note: You need to first check that $xR$ is a ring.]
(b) Suppose $x \in R$ is as above, i.e., $x^2 = x$ and $x$ commutes with every $r \in R$. Then show that $R \cong xR \times (1 - x)R$ as rings.
Let $R$ be a ring with identity. Assume all modules over it to be unitary and all ring homomorphisms to be unital.

(a) Let $M$ be an $R$-divisible module over $R$. If $N \subset M$ is an $R$-submodule of $M$, then show that the $R$-module $M/N$ is also $R$-divisible.

Now assume further that $R$ is commutative and let $P \subset R$ be a nonzero prime ideal. Set $R_P = S^{-1}R$, where $S = R \setminus P$.

(b) Define the $R_P$-module homomorphisms $i, j$ in the sequence

$$0 \to PR_P \xrightarrow{i} R_P \xrightarrow{j} R_P/(PR_P) \to 0,$$

so that it becomes a short exact sequence of $R_P$-modules.

(c) Is the above short exact sequence a split exact sequence? Complete justification is needed. [Hint: You may use Kaplansky’s Theorem: Any projective module over a commutative local ring, which is unital by definition, is free.]
Let $R$ be a commutative and unital UFD. Let $f(x) \in R[x]$ be a polynomial.

(a) Choose any $\alpha \in R$. Is it true that $f(x)$ is irreducible in $R[x]$ if and only if $f(x - \alpha)$ is irreducible in $R[x]$? Complete justification is needed.

(b) Either prove the following assertion or provide a counter-example:
   If a primitive polynomial is reducible in $\mathbb{Z}[x]$ then so is it in $\mathbb{F}_p[x]$, where $p$ is a prime number and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

(c) Is the polynomial $x^2 + x + 1$ irreducible in $\mathbb{Z}[x]$? Complete justification is needed.
(5) [3+2+2+3 points]

Let $R$ be a ring. Consider $\tilde{R} = R \oplus \mathbb{Z}$ as an abelian group. Define a multiplication on $\tilde{R}$ as $(r, n).(r', n') = (rr' + rn' + r'n, nn')$.

(a) Show that with the multiplication defined above $\tilde{R}$ is a unital ring. What is the identity element in $\tilde{R}$?

(b) Consider the injective ring homomorphism $i_R : R \rightarrow \tilde{R}$ sending $r \mapsto (r, 0)$. Show that $i_R(R)$ is a two-sided ideal in $\tilde{R}$.

(c) Let $f : R \rightarrow S$ be a ring homomorphism. Define the dotted ring homomorphism $\tilde{f}$ in the diagram below which makes it commute.

\[
\begin{array}{ccc}
R & \xrightarrow{f} & S \\
\downarrow{i_R} & & \downarrow{i_S} \\
\tilde{R} & \xrightarrow{\tilde{f}} & \tilde{S}
\end{array}
\]

Show that $\tilde{f}$ is unital.

(d) If $M$ is an $R$-module show that $M$ naturally becomes a unitary $\tilde{R}$-module such that if the module structure is restricted to $R$ via the ring homomorphism $i_R : R \rightarrow \tilde{R}$ it is isomorphic to the original $R$-module. [Hint: Try to find the dotted ring homomorphism in the diagram $R \xrightarrow{i_R} \text{End}_\mathbb{Z}(M)$ .]
(6) [5×2 points]
Either prove or provide a counter-example/argument to the following assertions:
(a) If \( f : R \to S \) is a nonzero ring homomorphism between unital rings then necessarily \( f(1_R) = 1_S \). [Recall the definition of a ring homomorphism from Problem (2).] What happens if \( S \) is a unital integral domain?
(b) Applying \(-\otimes \mathbb{Z} \mathbb{Q}\) to the short exact sequence of abelian groups \(0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0\) produces another short exact sequence of abelian groups with the canonical induced maps.
(c) \( \mathbb{C} \otimes_R \mathbb{C} \cong \mathbb{C} \otimes_\mathbb{C} \mathbb{C} \) as \( \mathbb{R} \)-vector spaces. Here \( \mathbb{C} \) is viewed as an \( \mathbb{R} \) module via the natural inclusion of fields \( \mathbb{R} \to \mathbb{C} \).
(d) Let \( F \) be a field. Let \((x)\) be the principal ideal generated by \( x \in F[x]\). Let \( K = S^{-1}(F[x]) \), where \( S = F[x] \setminus (x) \). Then \( K/(x)K \) is a field.
(e) Let \( F \) be a field. Then any vector space over \( F \) is reflexive.