(1) Let $R$ be a commutative integral domain with identity. Show that if $R$ viewed as an
$R$-module is injective then $R$ must be a field.

(2) What is $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ as an abelian group?

(3) Classify all finitely generated modules over a field. [Observe: A field is a commutative
and unital PID. However, you can do better than the classification result over a PID].

(4) Let $R$ and $S$ be commutative and unital rings and $\phi : R \to S$ be a unital ring
homomorphism. For any $s_1, \ldots, s_n \in S$, show that there exists a unique ring homo-
morphism $\overline{\phi} : R[x_1, \ldots, x_n] \to S$ such that $\overline{\phi}(r) = \phi(r)$ for all $r \in R$ and $\overline{\phi}(x_i) = s_i$
for $i = 1, \ldots, n$.

The ring homomorphism $\overline{\phi}$ is called the substitution or evaluation homomorphism.

(5) Let $R$ be a unital ring and let $S_n$ denote the symmetric group on $n$ letters. Given
$\sigma \in S_n$ define a map $\theta_{\sigma} : R[x_1, \ldots, x_n] \to R[x_1, \ldots, x_n]$ by $\theta_{\sigma}(f(x_1, \ldots, x_n)) =
f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

(a) Show that $\theta_{\sigma}$ is a ring automorphism of $R[x_1, \ldots, x_n]$.

(b) Let $\text{Aut}(R[x_1, \ldots, x_n])$ denote the group of ring automorphisms of $R[x_1, \ldots, x_n]$.
Show that the map $S_n \to \text{Aut}(R[x_1, \ldots, x_n])$ sending $\sigma \mapsto \theta_{\sigma}$ is an injective
group homomorphism.

(c) Let $\phi : G \to \text{Aut}(R[x_1, \ldots, x_n])$ be any group homomorphism. Let us define
$R[x_1, \ldots, x_n]^{\phi(G)}$ to be $\{ f \in R[x_1, \ldots, x_n] \mid [\phi(g)](f) = f \ \forall \ g \in G \}$. Show that
$R[x_1, \ldots, x_n]^{\phi(G)}$ is a subring of $R[x_1, \ldots, x_n]$. 