1. Let \( R \) be a commutative ring with unity. Assume \( R \) is an injective \( R \)-module.

   - by comment in D.F. p. 347, \( R \) is an injective module over an ID is divisible
   - \( rR = R \) for all \( r \in R \setminus \{0\} \)
   - for any nonzero \( a \in R \), \( 3r \in R \setminus \{0\} \) s.t. \( ra = 1 \), i.e. \( a \in R^\times \)
   - \( R \) is a field

2. Consider the SES:
   \[
   0 \to Z \to Q \to Q\otimes Z \to 0
   \]

   Apply the right exact functor \( Q_{\otimes } - \):

   \[
   Q_{\otimes } Z \to Q_{\otimes } Q \to Q_{\otimes } Q/2 \to 0
   \]

   Since \( Q_{\otimes } Z \cong Q \), and \( Q \to Q_{\otimes } Q \), the sequence is exact on the left (could also note \( Q \) is \( Z \) localized at \( (0) \) is flat)

   Therefore, we have SES:

   \[
   0 \to Q_{\otimes } Z \to Q_{\otimes } Q \to Q_{\otimes } Q/2 \to 0
   \]

   \[
   0 \to Q_{\otimes } Q \to Q_{\otimes } Q \to Q_{\otimes } Q/2 \to 0
   \]

   by exactness, \( Q \cong Q_{\otimes } Q \)

3. The only proper ideals in a field is \( (0) \), since every element is a unit. Thus a field is trivially a PID.

   By the FTFG MOPID, any \( R \)-module \( M \cong R \oplus \oplus R \oplus \oplus \oplus R \oplus \oplus \oplus R \) for \( \epsilon \neq 0 \)

   so every module over a field is torsion-free, which over a PID is free of rank \( \epsilon \) (\( \epsilon \) dim \( \text{dim}_R \)

4. For \( i = 1, \ldots, n \). For any polynomial \( f(x) = \sum x_i \), \( \overline{f}(x) = \sum \overline{x_i} \) \( \overline{x_i} = x_i \cdot \overline{1} = x_i \cdot \overline{1} \) .

   So \( \overline{f}(x) \) is completely determined by the images of the \( x_i \)‘s, which provides uniqueness. \( \phi \) clearly a ring homomorphism.

   For the diagram:

   \[
   \begin{array}{ccc}
   R & \to & R[x_1, \ldots, x_n] \\
   \phi & \downarrow & \downarrow \\
   \overline{\phi} & \to & \overline{R}[\overline{x_1}, \ldots, \overline{x_n}]
   \end{array}
   \]

   to commute, \( \overline{\phi} \mid_R = \phi \) as well.

5. a) Since \( S_n \) is a group, for any \( \sigma \in S_n \), \( \sigma^{-1} \), so the inverse of \( \sigma \) is just \( \sigma^{-1} \).

   b) The action is transitive, so the only \( \sigma \in S_n \) s.t. \( \sigma \to \text{id} \in \text{Aut}(R[x_1, \ldots, x_n]) \)

   c) Easy to check this is a subgroup. Note for \( G \subseteq S_n \), the fixed elements \( R[x_1, \ldots, x_n]^G \) are the symmetric polynomials.

   (and \( \phi = 0 \))