Integration by parts.

Differential rules → Integration rules.

Recall product rule:

\[
\frac{d}{dx} [f(x)g(x)] = f(x)g'(x) + g(x)f'(x)
\]

take indefinite integrals on both sides:

\[
f(x)g(x) = \int f(x)g'(x) \, dx + \int g(x)f'(x) \, dx
\]

rearrange

\[
\int f(x)g'(x) \, dx = \frac{f(x)g(x)}{u} - \frac{\int g(x)f'(x) \, dx}{v} \frac{dv}{du}
\]

Let \( u = f(x) \) \( v = g(x) \) \( du = f'(x) \, dx \) \( dv = g'(x) \, dx \)

\[
\int u \, dv = uv - \int v \, du
\]
Example 1: \( \int x \cos 2x \, dx \)

Solution: \( u = x \quad dv = \cos 2x \, dx \)

Then \( du = dx \quad v = \frac{1}{2} \sin 2x + C \), but we choose \( c = 0 \)

Integration by parts gives

\[
\int x \cos 2x \, dx = \frac{1}{2} x \sin 2x - \int \frac{1}{2} \sin 2x \, dx
\]

\[
= \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + C
\]

Alternate solution:

\( u = \cos 2x \quad dv = x \, dx \)

\( du = -2 \sin 2x \, dx \quad v = \frac{1}{2} x^2 \)

\[
\int x \cos 2x \, dx = \frac{1}{2} x^2 \cos 2x - \int \frac{1}{2} x^2 \cdot (-2) \sin 2x \, dx
\]

more complicated

Integration by parts twice (or more).

\[
\int t^3 e^t \, dt = u = t^3 \quad dv = e^t \, dt
\]

\( du = 3t^2 \, dt \quad v = e^t \)

So \( \int t^3 e^t \, dt = t^3 e^t - \int 3t^2 e^t \, dt \)

simpler! keep doing the same trick!
General idea of making choices.

Let $u$ be the one whose derivative becomes simpler.

Example 2. $\int t^2 e^t \, dt$

Choices:
1. $u = t^2$
   \[ dv = e^t \, dt \]
   \[ v = e^t \]
2. $u = e^t$
   \[ dv = t^2 \, dt \]

Choice 1 is better since $du = 2t \, dt$ is simpler.

Choice 2 will have the same issue as in 1.

Let us do 1:

\[ \int dv = 2t \, dt \]

\[ \int t^2 e^t \, dt = t^2 e^t - 2 \int t e^t \, dt \]

What about now? Do it again!

\[ \int t e^t \, dt = t e^t - \int e^t \, dt \]

\[ = (t-1) e^t + C \]

Combining them together:

\[ \int t^2 e^t \, dt = t^2 e^t - 2 (t-1) e^t - 2C \]

\[ = (t^2 - 2t + 2) e^t + C_1 \]
Eg 3: \( \int e^x \cos x \, dx \).

\[
\begin{align*}
\text{try } (0): & \quad \int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx \\
\text{did not improve anything, but do it one more time:} & \\
\int e^x \sin x \, dx & = \int e^x \, d(\sin x) \\
& = e^x \sin x - \int (\cos x) e^x \, dx \\
& = e^x \sin x + \int e^x \cos x \, dx
\end{align*}
\]

\[
2 \int e^x \cos x \, dx = e^x (\sin x + \cos x)
\]

\[
\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x)
\]

Most of the time, any choice is not preferable before analysis.

But we learn by experience. There are certainly more than one routes for one problem.
Definite Integrals.

\[
\int_{a}^{b} f(x) g'(x) \, dx = f(x) g(x) \bigg|_{a}^{b} - \int_{a}^{b} f(x) f'(x) \, dx
\]

eg 4. \[\int_{0}^{1} \tan^{-1} x \, dx\]

Solution:

\[
\begin{align*}
    u &= \tan^{-1} x & du &= \frac{dx}{1 + x^2} \\
    dv &= dx & v &= x
\end{align*}
\]

\[
\int_{0}^{1} \tan^{-1} x \, dx = \tan^{-1} x \cdot x \bigg|_{0}^{1} - \int_{0}^{1} \frac{x}{1 + x^2} \, dx
\]

\[
= \tan^{-1} 1 - \tan^{-1} 0 \cdot 0 - \int_{0}^{1} \frac{x}{1 + x^2} \, dx
\]

Substitution \( t = 1 + x^2 \), \( dt = 2x \, dx \)

\[
= \frac{\pi}{4} - \frac{1}{2} \int_{1}^{2} \frac{1}{t} \, dt
\]

\[
= \frac{\pi}{4} - \frac{1}{2} \ln t \bigg|_{1}^{2} = \frac{\pi}{4} - (\frac{1}{2} \ln 2 - \frac{1}{2} \ln 1)
\]

\[
= \frac{\pi}{4} - \frac{1}{2} \ln 2.
\]

So \( \int_{0}^{1} \tan^{-1} x \, dx = \frac{\pi}{4} - \frac{1}{2} \ln 2. \)
eg5. \[ \int \sin^2 x \, dx \]

Remember
\[
\begin{align*}
sin 2x &= 2 \sin x \cos x \\
\cos 2x &= \cos^2 x - \sin^2 x
\end{align*}
\]

Try
(1) \[
\begin{align*}
u &= \sin^2 x \\
du &= 2\sin x \cos x \, dx
\end{align*}
\]
\[
\begin{align*}
v &= \cos x \\
dv &= \cos x \, dx
\end{align*}
\]

(2) \[
\begin{align*}
u &= \sin x \\
du &= \cos x \, dx
\end{align*}
\]
\[
\begin{align*}
v &= -\cos x \\
dv &= \cos x \, dx
\end{align*}
\]

Try 2. \[
\int \sin^2 x \, dx = \sin x (-\cos x) - \int (-\cos x) \cos x \, dx
\]
\[
= -\sin x \cos x + \int \cos^2 x \, dx
\]
\[
= -\sin x \cos x + \int (1 - \sin^2 x) \, dx
\]
\[
2 \int \sin^2 x \, dx = x - \sin x \cos x + C
\]
\[
\int \sin^2 x \, dx = \frac{1}{2} (x - \sin x \cos x) + C_1
\]

Do it again for \( \int \cos^2 x \, dx \).

\[
\begin{align*}
u &= \cos x \\
du &= -\sin x \\
dv &= \cos x \, dx \\
v &= \sin x
\end{align*}
\]

\[ \int \cos^2 x \, dx = \cos x \sin x + \int \sin^2 x \, dx \]

Does not help
Summary on Integration by parts

\[ \int u \, dv = uv - \int v \, du \]

for \( u, v \) differentiable function.

Remark ① making decisions on \( u \) and \( v \) is essential in using formula.

② Valid for definite integrals as well.

③ For indefinite integrals, the "constant" appearing in the final answer plays a role of eliminating ambiguity.

Don't think too much about it.

Section 7.2: Trigonometric Integrals

Review ① \( \sin^2 \theta + \cos^2 \theta = 1 \)

② \( \sin 2\theta = 2 \sin \theta \cos \theta \)

③ \( \cos 2\theta = \cos^2 \theta - \sin^2 \theta \)

\[ = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta \]

half-angle formula \( 1 + \cos 2\theta = 2 \cos^2 \theta \Rightarrow 1 + \cos \theta = 2 \cos^2 \frac{\theta}{2} \)

④ \( \sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \)
\[ \cos(a + b) = \cos a \cos b - \sin a \sin b \]

Use (5) to deduce (6) & (7)

6. \[ \sin a \cos b = \frac{1}{2} \left[ \sin(a - b) + \sin(a + b) \right] \]

7. \[ \sin a \sin b = \frac{1}{2} \left[ \cos(a - b) - \cos(a + b) \right] \]

Goal for today: \[ \int \sin^m x \cos^n x \, dx \] when \( m, n \) positive integers.

Two cases:
1. Either \( m \) or \( n \) is odd.
2. \( m \) and \( n \) are both even

Example 1: \[ \int \sin^3 x \cos x \, dx \]

One strategy is to use \( \sin^2 x + \cos^2 x = 1 \) to rewrite the integral in the form:

\[ \int \sin^3 x \cos^2 x \, dx = \int f(\cos x) \sin x \, dx \]

Note that

\[ \sin^3 x \cos^2 x = \sin^2 x \cos^2 x \sin x = (1 - \cos^2 x) \cos^2 x \sin x. \]

Next use substitution
\[ u = \cos x \quad \text{and} \quad du = -\sin x \, dx. \]

Then
\[
\int \sin^3 x \cos^3 x \, dx = \int (1-\cos^2 x) \cos^2 x \sin x \, dx
\]
\[
= \int (1-u^2) u^2 (-du)
\]
\[
= \int (u^4-u^2) \, du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = -\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + C
\]

Eq. 2:  \[ \int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx \]
\[
= \int (1-\sin^2 x) \cos x \, dx
\]
\[
u = \sin x \quad \Rightarrow \quad du = \cos x \, dx
\]
\[
= \int (1-u^2) \, du = \int 1 - 2u^2 + u^4 \, du
\]
\[
= u - 2 \cdot \frac{u^3}{3} + \frac{u^5}{5} + C = \frac{\sin^5 x}{5} - 2 \cdot \frac{\sin^3 x}{3} + \sin x + C
\]

Case 2. (both \( m \) and \( n \) are even)

Need to use double angle formula such as
\[
\cos^2 x = \frac{1}{2} (1+\cos 2x)
\]
\[
\sin^2 x = \frac{1}{2} (1-\cos 2x)
\]
eq 3. \[ \int \cos^2 x \, dx = \int \frac{1}{2} (1+\cos 2x) \, dx \]

\[ = \frac{1}{2} x + \frac{1}{4} \sin 2x + C. \]

Similarly, \[ \int \sin^2 x \, dx = \int \frac{1}{2} (1-\cos 2x) \, dx \]

\[ = \frac{1}{2} x + \frac{\sin 2x}{4} + C. \]

General strategy for case 2 is to apply double angle formula until we came back to case 1.

eg 4. \[ \int \sin^2 x \cos^2 x \, dx \]

Applying the method twice yields

\[ \int \frac{1-\cos 2x}{2} \cdot \frac{\sin 2x}{2} \, dx \]

\[ = \int \left( \frac{1}{4} - \frac{1}{8} \cos 4x \right) \, dx \]

\[ = \frac{1}{8} x - \frac{1}{32} \sin 4x + C \]
There is a short cut for eq 4

\[ \sin 2x = 2 \sin x \cos x \]

\[
\int \sin^2 x \cos^2 x \, dx = \int (\frac{1}{2} \sin 2x)^2 \, dx = \frac{1}{4} \int \frac{1-\cos 4x}{2} \, dx
\]

= same as above

\[ \int \sec^n x \tan^m x \, dx \quad m, n \geq 0 \]

Recall: \[ \sec^2 x = 1 + \tan^2 x \]

\[ \int \sec x \, dx = \tan x + C. \]

\[ \int \sec x \tan x \, dx = \sec x + C \]

two cases with clear strategies

power of \( \tan x \) is odd

power of \( \sec x \) is even.
eg5. \[ \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx. \]

Use substitution

\[ u = \cos x \quad du = -\sin x \, dx \]

then

\[ \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int -\frac{du}{u} = -\ln |u| + C \]

\[ = -\ln |\cos x| + C \]

\[ \square \]

eg6: \[ \int \sec^4 x \, dx = \]

We have a factor \( \sec^2 x \), and rewrite the remaining as a function of \( \tan x \).

\[ \sec^4 x = \sec^2 x \cdot \sec^2 x = (1 + \tan^2 x) \cdot \sec^2 x \]

Now use substitution.

\[ u = \tan x \quad du = \sec^2 x \, dx \]

\[ \int \sec^4 x \, dx = \int (1 + u^2) \, du = u + \frac{u^3}{3} + C = \tan x + \frac{1}{3} \tan^3 x + C \]

\[ \square \]
eg7. \( \int \tan^3 x \sec x \, dx \)

Save a factor of \( \tan x \sec x \), use \( \tan^2 x = \sec^2 x - 1 \)

\[
\tan^3 x \sec x = (\sec^2 x - 1) \tan x \sec x
\]

\[
\int \tan^3 x \sec x \, dx = \int (\sec^2 x - 1) \tan x \sec x \, dx
\]

\[
= \int (\sec^2 x - 1) \, d\sec x
\]

Let \( u = \sec x \), \( du = \sec x \tan x \, dx \)

Then \( \int \tan^3 x \sec x = \int (u^2 - 1) \, du \)

\[
= \frac{u^3}{3} - u + C
\]

\[
= \frac{\sec^3 x}{3} - \sec x + C
\]

eg8. \( \int \sec x \, dx \)

Easier solution but harder to come up with

\( u = \tan x + \sec x \) \quad \( du = (\sec^2 x + \sec x \tan x) \, dx \)

Then
\[ \int \sec x \, dx = \int \sec x \cdot \left( \frac{\sec x + \tan x}{\sec x + \tan x} \right) \, dx \]

\[ = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \]

\[ = \int \frac{du}{u} \]

\[ = \ln |u| + C = \ln |\tan x + \sec x| + C \]

*The natural solution, but harder to proceed w/\*

\[ \int \sec x \, dx = \int \cos^{-1} x \, dx \]

\[ = \int \frac{\cos x}{\cos^2 x} \, dx \]

\[ = \int \frac{\cos x}{1 - \sin^2 x} \, dx \]

\[ u = \sin x \quad du = \cos x \, dx \]

\[ = \int \frac{du}{1 - u^2} = \frac{1}{2} \left( \int \frac{1}{1 + u} \, du + \int \frac{1}{1 - u} \, du \right) \]

\[ = \frac{1}{2} \left( \ln |u + 1| - \ln |u - 1| \right) + C \]

\[ = \frac{1}{2} \ln \left| \frac{\sin x + 1}{\sin x - 1} \right| + C \]