Chapter 9  Differential Equations.

Def: A differential equation (DE) is an equation that contains an unknown function and one or more of its derivatives.

Order of DE: the highest derivative's order.

Eq.  \( y' = xy \) (*) \( y = y(x) \) is the unknown function
\[ \text{order} = 1 \]
\[ y'' + y = x \quad \text{order} = 2. \]

(2) A function \( f \) is called a solution of a DE if the equation is satisfied when \( y = f(x) \) and its derivatives are substituted into the equation.

Eq.  \( f \) is solution of (*) if \( f'(x) = xf(x) \).
\[ f(x) = \frac{x^4}{4} + c \] is a solution to \( y' = x^3 \).

Eq.  Show that \( y = ce^x + e^{2x} \) for every constant \( c \) is a solution to \( y' - y = e^{2x} \).
proof: For any constant $c$, $y = ce^x + e^{2x}$
then $y' = ce^x + 2e^{2x}$

the LHS = $y' - y = e^{2x}$ = the RHS of DE. \(\square\)

Initial value problem
Find a solution to the differential equation satisfying

$y(x_0) = y_0$.

eg: Find a solution to $y' - y = e^{2x}$ st $y(0) = 1$.

Substitute the value $x = 0$ and $y = 1$ into

$y = ce^x + 2e^{2x}$, we get

$1 = c + 2$

$c = -1$

So

$y = 2e^{2x} - e^x$
First order differential equation.

\[ y' = F(x, y) \] \quad \text{if \ } F(x, y) \text{ is some expression in } x \text{ and } y \]

It is called separable if \( F(x, y) = g(x)f(y) \)

Solve separable DE.

If \( f(y) \neq 0 \), we could write:

\[ \frac{dy}{dx} = g(x)f(y) \]

\[ \int \frac{1}{f(y)} \, dy = \int g(x) \, dx \]

then solve \( y \).

**e.g.:** (a) \( \frac{dy}{dx} = \frac{x^2}{y^2} \)  \quad (b) Find the solution with \( y(0) = 2 \).

\[ y^2 \, dy = x^2 \, dx \Rightarrow \int y^2 \, dy = \int x^2 \, dx \]

\[ \Rightarrow \frac{y^3}{3} = \frac{x^3}{3} + C \Rightarrow y^3 = x^3 + 3C \Rightarrow y = \sqrt[3]{x^3 + 3C} \]

(b) put \( x = 0 \) and \( y = 2 \) into \( y = \sqrt[3]{x^3 + 3C} \)

we get \( 3C = 8 \). So \( y = \sqrt[3]{x^3 + 8} \)
Models for population growth.

\[ x : \text{time variable} \]

\[ P(x) : \text{the population at time } x \]

\[ P'(x) : \text{the growth rate at time } x \]

Law: the growth rate is proportional to \( P(x) \)

1000 bacteria growing at rate 300 bacteria.

2000 bacteria growing at rate 600 bacteria.

\[
\frac{dP(x)}{dx} = kP(x) \quad \Leftrightarrow \quad \frac{dy}{dx} = ky
\]

Eg: (a) Solve \( \frac{dy}{dx} = ky \)  
(b) Find the solution s.t. \( y(1) = 2 \)

\[
\frac{dy}{y} = kdx \quad \Rightarrow \quad \int \frac{dy}{y} = \int kdx \quad \Rightarrow \quad \ln|y| = kx + C
\]

\[
\Rightarrow \quad y = e^{kx} \cdot e^C = C_1 e^{kx}
\]

(b). Plug \( x=1 \) and \( y=2 \) into

\[
y = C_1 e^{kx}
\]

\[
C_1 = \frac{2}{e^k} \quad \Rightarrow \quad y = 2e^{kx} + k
\]
Linear equations with first order.

\[
\frac{dy}{dx} + p(x)y = q(x)
\]

where \( p(x) \) and \( q(x) \) are continuous functions on a given interval.

Strategy to solve: Multiply both sides by \( I(x) = e^{\int p(x) \, dx} \) and integrate both sides.

\[ (\text{Solve}) \]

Eg.: \( y' - y = e^x \). Linear, \( p(x) = -1 \) \( q(x) = e^x \).

\[ I(x) = e^{\int p(x) \, dx} = e^{-x} \]

\[ e^{-x} y' - e^{-x} y = 1 \iff \frac{d(e^{-x} y)}{dx} = 1 \]

\[ e^{-x} \frac{dy}{dx} - e^{-x} y = 1 \]

\[ \Rightarrow \text{Integrating on both sides:} \]

\[ e^{-x} y = x + C \]

\[ y = e^x \cdot (x + C e^x) \]
(b) Find the solution with $y(1)=2$

plug $x=1$ and $y=2$ into

$$y = e^x \cdot x + ce^x$$

$$2 = e + ce \quad c = \frac{2}{e} - 1$$

so

$$y = e^x \cdot x + \left(\frac{2}{e} - 1\right)e^x$$