1) (a) To see if \( p_1(x), p_2(x), p_3(x) \) are linearly dependent, we check if the linear system

\[
\begin{bmatrix}
1 & -1 & 1 & 0 \\
-1 & 1 & -1 & 0 \\
0 & 1 & 1 & 0 \\
2 & -3 & 1 & 0
\end{bmatrix}
\]

admits non-trivial solutions. By Gaussian elimination, (*) can be reduced to

\[
\begin{bmatrix}
1 & -1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

which leads to \( 2p_1(x) + p_2(x) = p_3(x) \).

Hence \( p_1(x), p_2(x), p_3(x) \) are linearly dependent.

Since \( p_1(x), p_2(x) \) are not multiples, they are linearly independent, and therefore they form a basis for \( \text{span} \{ p_1(x), p_2(x), p_3(x) \} \).
(b) \( p_1(x), p_4(x) \) are not multiples, so they are linearly independent.

Since \( \{1, x, x^2, x^3\} \) is a basis of \( P_3(\mathbb{R}) \), there it contains two vectors that complete \( \{ p_1(x), p_4(x) \} \) to a basis of \( P_3(\mathbb{R}) \).

Performing Gaussian elimination on

\[
\begin{bmatrix}
1 & 3 & 0 & 0 & 0 \\
-1 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
2 & -1 & 0 & 0 & 0
\end{bmatrix}
\]

we obtain that \( p_1(x), p_4(x), x^2, x \) are linearly independent, and therefore they form a basis of \( P_3(\mathbb{R}) \).

2) \( \implies \) Suppose \( W \) is a \( m \)-dim subspace of \( V \). Take a basis \( w_1, \ldots, w_m \) of \( W \) and define the linear map

\( T : \mathbb{R}^n \rightarrow V \) such that \( T(e_i) = w_i \) for all \( i = 1, \ldots, n \), where \( e_1, \ldots, e_n \) is the standard basis of \( \mathbb{R}^n \).

By the dimension theorem, we have

\( \text{dim ker } T = 0 \), hence \( \ker T = \{0\} \), that is, \( T \) is injective.

(\( \implies \)) Suppose we have a linear injection \( T : \mathbb{R}^n \rightarrow V \). Take \( W = \text{ran } T \). Then \( W \) is \( m \)-dim by the dimension theorem.
3) (a) \[ V = \{ (x_1, x_2, x_3, x_2-x_1, x_3) \mid x_1, x_2, x_3 \in \mathbb{R} \} \]

A basis for \( V \) is
\[ \{(1, 0, 0, -1, 0), (0, 1, 0, 1, 0), (0, 0, 1, 0, 1)\} \]
\[ \dim V = 3. \]

A basis for \( W \) is given by
\[ (1, 0, 0, 0, 0), (0, 1, 1, 1, 1), \]
\[ \dim W = 2. \]

(b) Since \((1, 0, 0, 0, 0) \in V \) and \((0, 1, 1, 1, 1) \in V\),
\[ \text{then } V \cap W = \text{span} \{(0, 1, 1, 1, 1)\} \]
\[ \text{and } \dim V \cap W = 1. \]

Thus, \[ \dim V + W = \dim V + \dim W - 1 \]
\[ = 4. \]

4) \((\Rightarrow)\) Suppose \( T: V \rightarrow W \) is injective.
Let \( v_1, \ldots, v_n \in V \) be linearly independent and take \( T v_1, \ldots, T v_n \).
Let \( \alpha_1 T v_1 + \cdots + \alpha_n T v_n = 0 \) for some \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \).
Then, by linearity, \( T (\alpha_1 v_1 + \cdots + \alpha_n v_n) = 0 \).
Since \( T \) is injective, \( \alpha_1 v_1 + \cdots + \alpha_n v_n = 0 \),
which implies \( \alpha_1 = \cdots = \alpha_n = 0 \).
Therefore \( T v_1, \ldots, T v_n \) are linearly independent.

\((\Leftarrow)\) Suppose \( T: V \rightarrow W \) maps line. indep. sets into line. indep. sets. Let \( v \in V, \ v \neq 0 \). Then \( \{v\} \) is line. indep. and \( T \{v\} = \{T v\} \) must also be line. indep., that is, \( T v \neq 0 \).
5) (a) \( T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \ T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ T \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

\[ [T]_\alpha = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix}, \ \alpha = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \} \]

(b) \( T(x, y, z) = 0 \) yields \( x = y = z = 0 \),

Hence \( \ker T = \{ 0 \} \), and \( \text{ran } T = \mathbb{R}^3 \)

by the dimension theorem.

(c) Since \( T \) is injective,

\( \beta := T(\alpha) \) is a basis for \( \mathbb{R}^3 \),

thus \( [T]_\beta^\alpha = I_3 \).

(d) Let \( \alpha \) be a basis for \( V \).

Since \( T \) is injective and \( \dim W = \dim V \),

\( \beta := T(\alpha) \) is a basis for \( W \)

and \( [T]_\beta^\alpha = I_n \).