1) The coefficient matrix is

$$A_K = \begin{bmatrix} 1 & K & 2 \\ K & 1 & -K \\ 2 & 2K & 1 \end{bmatrix}$$

whose determinant is

$$\det(A_K) = 1 + 2K^2 - K(K+2K) + 2(2K^2-2) = 3(K-1)(K+1).$$

For $K \in \mathbb{R}\setminus\{\pm 1\}$, $\det(A_K) \neq 0$, hence the system has one solution, which can be computed by Gaussian elimination or Cramer rule ($\cdots$).

For $K = 1$, we have

$$A_1 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -1 \\ 2 & 2 & 1 \end{bmatrix},$$

which has rank 2 because $\det\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \neq 0$. On the other hand, the rank of the augmented matrix is also 2 because
\[
\det \begin{bmatrix}
1 & 2 & 1 \\
1 & -1 & 0 \\
2 & 1 & 1
\end{bmatrix} = 0,
\]

Therefore the system has a unique solution.

By Gaussian elimination (---), the set of solutions is
\[
\left\{ \left( t, \frac{1}{3} - t, \frac{1}{3} \right) \mid t \in \mathbb{R} \right\}.
\]

For \( \kappa = -1 \), \( A_{-1} = \begin{bmatrix} -1 & -1 & 2 \\ -1 & 1 & 1 \\ 2 & -2 & 1 \end{bmatrix} \) has rank 2 since
\[
\det \begin{bmatrix}
-1 & 2 \\
-1 & 1 \\
2 & -2
\end{bmatrix} \neq 0,
\]
but the rank of the augmented matrix is 3 since
\[
\det \begin{bmatrix}
-1 & 2 & -1 \\
-1 & 1 & 0 \\
2 & -2 & 1
\end{bmatrix} \neq 0,
\]
therefore the system has no solutions.

2) Suppose \( T \) is injective and let \( x \in V^* \).
Fix a basis \( v_1, \ldots, v_n \) for \( V \). Then \( w_1 = Tv_1, \ldots, w_n = Tv_n \) are linearly independent vectors in \( W \) since \( T \) is injective.
Let us complete \( w_1, \ldots, w_n \) to a basis \( w_1, \ldots, w_n, w_{n+1}, \ldots, w_m \) for \( W \), and define
\[0(\omega_i) = \alpha(\nu_i) \quad i = 1, \ldots, n\]
\[0(\omega_j) = \theta \quad j = n+1, \ldots, m\]
\[\beta \in W^* \text{ and } T^*\beta(\nu_i) = \beta T(\nu_i) = \beta(\omega_i) = \alpha(\nu_i)\]
for all \(i = 1, \ldots, n\), thus \(\alpha = T^*\beta\).  

Conversely, suppose \(T^*\) is surjective and let \(Tv = \theta\). Then, for all \(\beta \in W^*\)
\[0 = \langle Tv, \beta \rangle = \langle v, T^*\beta \rangle\]
thus, since \(T^*\) is surjective,
\[\langle v, \beta \rangle = \theta \quad \text{for all } \beta \in V^*, \]
whence \(v = \theta\).

3) The matrix representation of \(T_K\) with respect to the standard basis of \(\mathbb{R}^3\) is
\[
A_K = \begin{pmatrix}
1 & 0 & k^2 \\
0 & k & 0 \\
1 & 0 & 1
\end{pmatrix}
\]
The characteristic polynomial of \(T_K\) is
\[p(x) = \det(xI - A_K) = (x-k)(x-1-k)(x-1+k)\]
whose roots are \(k, 1+k, 1-k\).  
For \(k \neq 0, \frac{1}{2}\) the roots are all distinct, thus \(T_K\) is diagonalizable.  
For \(k = 0\), we have two eigenvalues,  
\[0 \text{ with algebraic multiplicity } 1 \text{ and } 1 \text{ with algebraic multiplicity } 2.\]
Now, \( m_\alpha(0) = 1 \) and hence \( m_\gamma(0) = 1 \) as well. To determine \( M_\gamma(1) \), we compute the rank of \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

Since \( \det \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \neq 0 \), \( \operatorname{rk} (\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) = 2 \), thus \( m_\gamma(1) = \dim V - \operatorname{rk} (\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) = 3 - 2 = 1 \neq m_\alpha(1) \) and therefore \( T_\theta \) is not diagonalizable.

For \( \lambda = \frac{1}{2} \), we have two eigenvalues, \( \frac{1}{2} \) with algebraic multiplicity 2 and \( \frac{3}{2} \) with alg. mult 1.

\( m_\gamma(\frac{3}{2}) = m_\alpha(\frac{3}{2}) = 1 \), while

\( m_\gamma(\frac{1}{2}) = 3 - \operatorname{rk} (\frac{1}{2} I - A_{\frac{1}{2}}) = 3 - 1 = 2 = m_\alpha(\frac{1}{2}) \), therefore \( T_{\frac{1}{2}} \) is diagonalizable.

4) \( \beta = \{v_1, \ldots, v_n\} \) be a basis of \( V \) s.t.

\[
[T]_\beta = \begin{bmatrix}
\lambda_1 & 0 \\
0 & -\lambda_2 \\
\end{bmatrix}, \quad \lambda_i \geq 0 \quad i = 1, \ldots, n.
\]

Define \( S v_i := \sqrt{\lambda_i} v_i \). Then

\[
[S]_\beta [S]_\beta = \begin{bmatrix}
\sqrt{\lambda_1} & 0 \\
0 & \sqrt{\lambda_2} \\
\end{bmatrix} \begin{bmatrix}
\sqrt{\lambda_1} & 0 \\
0 & \sqrt{\lambda_2} \\
\end{bmatrix} = \begin{bmatrix}
\lambda_1 & 0 \\
0 & -\lambda_2 \\
\end{bmatrix} = [T]_\beta,
\]

where \( S^2 = T \). \( \Box \)
5) Assume for every $v \in V \setminus \{0\}$ there is $\lambda \in \mathbb{F}$ such that $Tv = \lambda v$.

Let $u, v \in V \setminus \{0\}$. Then either $u, v$ are linearly independent, or $v = \alpha u$ for some $\alpha \in \mathbb{F} \setminus \{0\}$.

If $u, v$ are linearly independent, then

\[ \begin{align*}
T(u + v) &= \lambda_{u+v}(u + v) \\
Tu + Tv &= \lambda_u u + \lambda_v v,
\end{align*} \]

\[(\lambda_{u+v} - \lambda_u)u + (\lambda_{u+v} - \lambda_v)v = 0\]

implies $\lambda_u = \lambda_{u+v} = \lambda_v$.

Similarly, if $v = \alpha u$, then

\[ \begin{align*}
T(\alpha u) &= \lambda_{\alpha u} u = \lambda u \\
\alpha Tu &= \alpha \lambda_u u,
\end{align*} \]

\[\alpha(\lambda_v - \lambda_u)u = 0\]

implies $\lambda_u = \lambda_v$.

Therefore $\lambda_v = \lambda_u$ for all $u, v \in V \setminus \{0\}$, and hence $T = \lambda I$ where $\lambda = \lambda_u$ for any $u \in V \setminus \{0\}$.

The converse is trivial. \(\blacksquare\)