1) Let $V = \text{span} \{ p_1(x), p_2(x), p_3(x) \} \subseteq P_3(\mathbb{R})$
where $p_1(x) = 1 + 2x + 2x^2 + x^3$
$p_2(x) = 2x + x^3$
$p_3(x) = -2 - 4x^2 + 3x^3$

(a) Prove that $p_1(x), p_2(x), p_3(x)$ form a basis for $V$.
(b) Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \in V$
Find the components of $f(x)$ relative to the basis $p_1(x), p_2(x), p_3(x)$.
(c) Let $q_1(x) = 1 + 2x^2$, $q_2(x) = 2x + x^3$, $q_3(x) = 3x^3$
Show that $q_1(x), q_2(x), q_3(x)$ form a basis for $V$.

2) (a) State and prove a criterion for a linear transformation to be injective.
(b) Prove that a linear transformation maps linearly independent sets to linearly independent sets
if and only if it is injective.
(c) Prove that a linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ can not be injective if $m < n$. 
3) prove, or find a counterexample: any square matrix can be expressed uniquely as \( A = B + C \) with \( B^5 = B \) and \( C^5 = -C \).

4) let \( W = \text{span} \{(1, 2, 3, 10), (-2, 3, -4, 0), (-3, 4, -11, 9), (4, -5, 18, 1)\} \subseteq \mathbb{R}^4 \)

(a) find a basis for \( W \)
(b) find conditions for vectors \((a, b, c, d) \in \mathbb{R}^4\) to be in \( W \)

5) let \( V = P_2(\mathbb{R}) \) and \( T: V \to V \) a linear operator such that \( T(1) = x \), \( T(x) = 2x^2 \), \( T(x^2) = 3 \).

(a) write down the matrix of \( T \) under the basis \( 1, x, x^2 \).
(b) what is the matrix of \( T \) under the basis \( 1, x-2, (x-2)^2 \)?

6) let \( T: \mathbb{R}^4 \to \mathbb{R}^3 \) be the linear transformation whose matrix with respect to the standard basis is
\[
A = \begin{bmatrix}
1 & 0 & 3 & 1 \\
2 & -1 & 0 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}
\]

(a) find a basis for \( \text{ker} T \)
(b) find a basis for \( \text{ran} T \)
(c) determine \( T(7, -15, 0, 25) \)
7) Let $V_1, V_2$ be subspaces of a vector space $V$ let $T: V_1 \times V_2 \to V$ be the linear transformation defined by $T(v_1, v_2) = v_1 - v_2$

(a) show that $\ker T = \{0\} \iff V_1 \cap V_2 = \{0\}$

(b) show that $\dim(V_1 \cap V_2) \geq \dim V_1 + \dim V_2 - \dim V$

8) Prove or find a counterexample:
(a) if $V$ is a vector space and $W$ is a subset of $V$, then $W$ is a subspace of $V$
(b) subsets of linearly dependent sets are linearly dependent

9) State and prove the dimension theorem

10) Let $T: P_2(\mathbb{R}) \to \mathbb{R}^{2 \times 2}$ given by

$$T(p(x)) = \begin{bmatrix} p'(1) & p'(0) \\ p''(1) & p''(2) \end{bmatrix}$$

(a) Show that $T$ is linear
(b) Determine the rank of $T$
(c) Determine a basis for a subspace $W$ of $\mathbb{R}^{2 \times 2}$ such that $W \oplus \text{ran} T = \mathbb{R}^{2 \times 2}$
(d) Determine the matrix representation of $T$ with respect to the standard bases of $P_2(\mathbb{R})$ and $\mathbb{R}^{2 \times 2}$. 
11) Let $W$ be a subspace of $\mathbb{R}^n$ of dimension $n-1$ and show that there is a standard basis vector $e_i$ such that $\mathbb{R}^n = W + \text{span}\{e_i\}$.

12) Let $W_1$, $W_2$ be subspaces of $V$.
   (a) Define the sum of $W_1$ and $W_2$.
   (b) Prove, or find a counterexample:
       if $S_1$ is a basis for $W_1$ and $S_2$ is a basis for $W_2$, then $S_1 \cup S_2$ is a basis for $W_1 + W_2$.
   (c) Let $W_3$ be another subspace of $V$.
       define $W_1 \oplus W_2 \oplus W_3$.
   (d) Give an example of subspaces $W_1$, $W_2$, $W_3$ such that $V = W_1 + W_2 + W_3$, $W_1 \cap W_2 = W_1 \cap W_3 = W_2 \cap W_3 = \{0\}$ and $V \neq W_1 \oplus W_2 \oplus W_3$. 