JHU, FALL 2017
110.212 HONORS LINEAR ALGEBRA

REVIEW PROBLEMS (2)

1) Let $T: V \rightarrow V$ be a linear operator, and let $x \in V$. Suppose that $T^d x = 0$ but $T^{d-1} x \neq 0$.
Prove that $x, T x, T^2 x, \ldots, T^{d-1} x$
are linearly independent.

2) Let $V$ and $W$ be vector spaces of dimension $n$ and $m$, respectively. Find an isomorphism $L(V, W) \rightarrow M_{m \times n}$.

3) Let $T: V \rightarrow W$ be a linear mapping. Prove that $\ker (T^t) = (\text{ran} T)^0$.

4) Prove that every vector space is canonically isomorphic to its double dual.
5) Let $A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \in \mathbb{R}^{2 \times 2}$

(a) Determine a diagonal $2 \times 2$ matrix $D$ and an invertible $2 \times 2$ matrix $P$ for which $A = P^{-1}DP$

(b) Write $A$ as a product of elementary matrices.

6) Let $T: V \rightarrow V$ a diagonalizable operator. Suppose that $W$ is a proper $T$-invariant subspace of $V$. Show that $W$ has a $T$-invariant complementary subspace.

7) Let $M$ be a $2n \times 2n$ matrix in block form $\begin{pmatrix} A & B \\ \theta & C \end{pmatrix}$, where $A, B, C$ are $n \times n$ matrices.

Prove or disprove:

(a) $M$ is invertible if and only if $A$ and $C$ are invertible

(b) $M$ is diagonalizable when $A$ and $C$ are diagonalizable.
8) Discuss and solve the following linear systems depending on the real parameter $k$:

(a) \[
\begin{cases}
3x - y - kt = -1 \\
6x + ky + 4t = 1k
\end{cases}
\]

(b) \[
\begin{cases}
x + t + kt = 2 \\
x + y + 3t = k - 1 \\
2x + ky + t = 1
\end{cases}
\]

(c) \[
\begin{cases}
x + 2y + t = 0 \\
(k-1)y + t - t = k^2 + 3k \\
6t + (k+3)t = 0 \\
2x + 3y + t + 2t = 0
\end{cases}
\]

9) Let $T_{a}: \mathbb{R}^3 \to \mathbb{R}^3$, $T_{a}(x, y, t) = (x + ay + 3t, 2y + at, 4y + 2at)$

where $a \in \mathbb{R}$.

(a) For which values of $a$ is $T_{a}$ diagonalizable?

(b) Find a basis of eigenvectors of $T_{a}$ for $a = 0$.

(c) Compute $dim ker T_{a}$ and $rk(T_{a})$ for all $a \in \mathbb{R}$.
10) Let $T$ be a linear operator on a real vector space of dimension $n$. Suppose that $T-I$ is nilpotent (i.e. $(T-I)^m = 0$ for some $m \in \mathbb{N}$). Determine the eigenvalues and the characteristic polynomial of $T$.

11) Let $T: V \to V$ be a linear operator. Suppose $S: V \to V$ is linear and commutes with $T$. Show that each of the following subspaces of $V$ are $T$-invariant:

(a) $\text{ran}(S)$
(b) $E_{\lambda}(S)$ for any eigenvalue $\lambda$ of $S$
(c) $\ker((S - \alpha I)^2)$ for all $\alpha \in \mathbb{F}$

12) Let $T: V \to V$ be a diagonalizable operator on a real vector space $V$ with non-negative eigenvalues. Prove that there is a linear operator $S: V \to V$ such that $S^2 = T$. 