The alternating group A_n: simple for $n = 5$, not for $n = 4$

Why is it that A_4 has a PNT normal subgroup but A_n ($n \geq 5$) does not? We will try to answer this question as nicely as possible. Let me know if you find anything wrong with what follows.

0. First, the standard stuff about the full symmetric group S_n of permuations of $\{1, 2, \ldots, n\}$:

a) S_n is generated by 2-cycles (transpositions), i.e., every permutation is expressible as the product of a number of 2-cycles.

b) All r-cycles are conjugate: if σ and σ' are r-cycles, then there exists $\tau \in S_n$ with $\sigma' = \tau\sigma\tau^{-1}$ (see p. 150 or Exercises 1.4 #26 for the explicit formula).

c) Indeed, all products of disjoint r_1-, r_2-, ... , r_i-cycles are conjugate.

d) A_n is the kernel of the parity mapping, $\ker\{S_n \to \mathbb{Z}_2\}$, given by taking $\sigma \in S_n$ to the number of transpositions in #0 a).

1. *A_n is generated by 3-cycles.* This is true because A_n is certainly generated by products of two 2-cycles; by definition, its elements can be expressed as the product of an *even* number of 2-cycles. There are three cases: two equal 2-cycles, two disjoint 2-cycles, and two transpositions having one common element. We can argue by example since we know how to change the numbers (#0 b). We have respectively $(12)(12) = \epsilon$, $(12)(34) = (143)(123)$, $(12)(13) = (123)$. They are all expressed in terms of 3-cycles.

2. *All 3-cycles are conjugate in A_n.* Why? They are conjugate in S_n by #0 b); but we are now demanding more: that the conjugation be taken in A_n, i.e., by even permutations only, and that means we have half as many elements to conjugate by. First, it suffices to take conjugations by generators of A_n, for in any group, $(ab)g(ab)^{-1} = a(bgb^{-1})a^{-1}$. So by doing this repeatedly, we see that the general conjugate is given by iterated conjugation by the generators. Here, "generators" refers to 3-cycles, by #1.

We compute representative cases of aga^{-1}, using #0 b):

$(124)(123)(124)^{-1} = (243)$ — the rule is to do (124) to the numbers 1, 2, 3 — (a and g have two numbers in common),

$(145)(123)(145)^{-1} = (423)$ (a and g have one number in common),

$(456)(123)(456)^{-1} = (123)$ (a and g disjoint).

This shows how to replace one number by another. To get all 3-cycles from (123), we need do that up to three times.

3. *When $n \geq 5$, every normal subgroup of A_n contains a 3-cycle.* [See p. 151]. Thus it contains all 3-cycles by #2, therefore every even permutation by #1. In other words, A_n is simple for $n \geq 5$.

4. *In A_4, the set of all products of disjoint 2-cycles generates a proper normal subgroup.* This is because $[(12)(34)][(23)(14)] = (13)(24)$ (subgroup property) and $(123)(12)(34)(123)^{-1} = (23)(14)$ (normal property).

It is not hard to see that the order of this subgroup—call it K—is 4. Is it isomorphic to $\mathbb{Z}_4$, or is it $\mathbb{Z}_2 \times \mathbb{Z}_2$? How about identifying the factor group A_4/K up to isomorphism?