HW4: Outline of solutions to extra problems

2.3:

#17. *In a group, let H and K be subgroups. Show that*

$$H \cup K \text{ is a subgroup } \iff H \subseteq K \text{ or } K \subseteq H.$$

$\Leftarrow$) If $H \subseteq K$, then $H \cup K$ is just K, which is assumed to be a subgroup; if $K \subseteq H$, just reverse the roles of H and K.

$\Rightarrow$) Suppose $H \cup K$ is a subgroup, but both inclusions fail. Then there is an element $h \in H$ with $h \notin K$, and a $k \in K$ with $k \notin H$. But then both h and k are in $H \cup K$, and so is their product hk. This element must belong to H or K. If it's (say) H, $hk \in H$ implies $k \in hH = H$, a contradiction.

#22. *Determine the center of $GL_2(\mathbb{R})$.*

We are being asked to give the set of all invertible 2×2 matrices that commute with all invertible 2×2 matrices. One can solve for A the matrix equation that says $AX = XA$ for all X. Is it possible to do better than that? One course of action is to get started by seeing which matrices commute with

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and realize that's only the multiples of the identity matrix. The latter do commute with everything, so that's the answer.

2.4

#12. *If every proper subgroup of G is cyclic, must G be cyclic?*

No. $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a counterexample: all non-trivial proper subgroups are of order 2, hence cyclic.

#34. *Prove the Chinese remainder theorem.*

The best way to proceed, I think, is to turn the problem into a group-theoretic statement. One can say: show that the homomorphism of groups

$$\mathbb{Z} \to \mathbb{Z}_{n_1 \ldots n_r} \to \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_r}$$

is surjective under the given hypotheses on the n_i's. That is equivalent to the right-hand arrow's being a surjection. Since the two groups in question have the same number of elements, it's enough to show that this mapping is injective. Well, determine the kernel.