

Solution to 3.4 #46

Although this problem seems daunting, the issue is not that complicated at all. I will start by discussing it “hands on”. The reference to Theorem 7 says, in effect, that we’ve done this sort of thing already; we don’t have to start from scratch.

If R is any ring, the Gaussian ring $R(i)$ with coefficients in R is always isomorphic to $R \times R$ as an additive group. The mapping giving the isomorphism can be taken to be $\phi : R \times R \rightarrow R(i)$, given by $\phi(a, b) = a + bi$. But there are numerous other choices; this is like linear algebra. If we take, say, $i + i$ and $1 - i$, we do pretty well with $\phi'(a, b) = a(1 + i) + b(1 - i)$, though we don’t get the element $1 \in R(i)$ in the image unless $2 \in R^*$.

The mapping ϕ is not in general a homomorphism of rings; we were talking without regard to multiplication. Note that multiplication by i takes the “real” (i.e., first) factor of R to the “imaginary”, and vice versa. In a direct product, the factors must be preserved.

Let’s think about this as a problem with undetermined constants. By that, I mean: take two elements α and β of $R(i)$, and consider the corresponding mapping $\tilde{\phi} : R \times R \rightarrow R(i)$ as above, given by $\tilde{\phi}(a, b) = a\alpha + b\beta$. Here are some considerations.

- i) Factors preserved by left-multiplication — $\alpha r \alpha = s \alpha$ for some $s \in R$, and similarly for β
- ii) Factors preserved by right-multiplication : $\alpha^2 = s \alpha$ for some $s \in R$, and similarly for β
- iii) $\tilde{\phi}(1, 1) = 1 : \alpha + \beta = 1$
- iv) $\tilde{\phi}$ is one-to-one
- v) $\tilde{\phi}$ is onto.

Note that (i) and (ii) involve the ring structure, and the other three conditions are actually set-theoretic (*before* algebra).

If we are seeking **one** value for the pair (α, β) that works, you can try to thin out the possibilities, especially if you take the hint. You can knock off (i) and (ii) by saying, “let α and β be central idempotents.” We could be talking about the empty set, you know, so we have to produce such a pair. Why not $\alpha = \frac{1}{2}(1 + ui)$? (We do have that 2 is invertible in R .) Well, you’d better check that this is an idempotent:

$$\alpha^2 = \frac{1}{4}(1 + 2ui + u^2i^2) = \frac{1}{4}(2 + 2ui) = \alpha$$

and likewise for $\beta = \frac{1}{2}(1 - ui)$. These choices were concocted so that (iii) is obviously true. *Central?*

We’re left with (iv) and (v). This is about kernel and image. Does $\tilde{\phi}$ produce the value 0 in only the obvious way? Does it produce the value 1? Does it produce the value i ? Does that finish the problem?