Some more linear algebra

1. In what follows, V and W are vector spaces over the field F, and $T : V \to W$ a linear mapping. (You may take F to be the field of real numbers, if you wish, though I hope you will see that you do not need to make use of that fact.)

a) T is determined by its values on a basis of V. Here's another way of saying this. Let B be a basis of V. Then there is a one-to-one correspondence between mappings *of sets* $\phi : B \to W$ and linear mappings $T : V \to W$. If $B = \{v_1, ..., v_d\}$, the correspondence is

$$T_\phi(\sum_{i=1}^{d} c_i v_i) = \sum_{i=1}^{d} c_i \phi(v_i) \quad \text{(i.e., extend } \phi \text{ linearly)}$$

$$\phi_T = T|_B \quad \text{(i.e., restrict } T \text{ to the given basis)}$$

b) T is given by a matrix (in many ways). Do you remember how?

c) Suppose that $W = V$. Let A be the matrix of T with respect to some basis of V. What is the format of the matrix of T with respect to some other basis? (We will see this sort of expression a lot.)

2. Do matrices of size $d \times d$ form a field under addition and multiplication?

3. Let V be a vector space over F of dimension d. Then $V \simeq F^d$. That means one can define mutually inverse linear mappings $V \to F^d$ and $F^d \to V$. (These mappings are necessarily one-to-one and onto.) How is this done? *You might start with 1 a).*

It follows that any two vector spaces over F of dimension d are isomorphic. (Can we view isomorphism of vector spaces as an equivalence relation?)

4. a) Did we go through the verification that, for V a vector space and S a **subspace** of V, the relation

$$\text{For } v \text{ and } v' \text{ elements of } V,\ v' \equiv v \text{ if and only if } v' - v \in S$$

is an equivalence [relation] on V? Perhaps not. You can check that r (reflexivity), s (symmetry), and t (transitivity) all hold. All of these properties depend on features of *subspaces* (as opposed to arbitrary subsets) of V.

b) Likewise, one should check that the vector operations (addition and scalar multiplication) of V carry over to V/S, and this entails also checking that one's formulas are well-defined. To get started, given an equivalence class $[v]$ and a scalar c, the most natural way to do things is to put: $c \cdot [v] = [cv]$. We are predicating the definition on having an element of the equivalence class selected. *Who chose it? Whoever got there first?* You'll be imprisoned, powerless to maneuver, unless we get the same answer by this formula for other elements of $[v]$. Thus, we would need to know that $v' \equiv v$ implies $cv' \equiv cv$. See that this holds when S is a subspace of V. Checking addition is similar.