Solutions to the midterm exam, Nov. 1, 2004 90 minutes 100 pts.

There were 28 students taking the exam. First, some scores:

HIGH: 94, LOW: 29, MEDIAN: 67.

You must explain and justify your answers to all problems (except #1).

[10] 1. Give the definitions of the following notions:

a) the order of an element g in a group G;

b) a conjugate of an element $g \in G$.

[Find these in the textbook.]

[10] 2. In each case, give an example of a group G with $|G| \geq 10$ and:

a) the center of G equals G;

The question calls for an abelian group (every element is to commute with everything), so you can take $\mathbb{Z}_{10}$. Lots of choice, though!

b) the center of G is trivial.

The question calls for a group where only the identity element commutes with everybody. Lots of choice! A group with no PNT normal subgroups (i.e., a simple group) fits the bill: you can take A_5.

[20] 3. Let $f : \mathbb{Z} \to \mathbb{Z}_{50} \times \mathbb{Z}_2$ be the mapping defined by $f(x) = (\overline{x}_{50}, \overline{x}_2)$, where $\overline{x}_{50}$ is the residue class of x modulo 50, and $\overline{x}_2$ is the residue class of x modulo 2.

a) Show that f is a homomorphism of groups.

This may seem painfully obvious, but endure the pain. We know that the canonical mapping $\mathbb{Z} \to \mathbb{Z}_n$ is a homomorphism for all $n \in N$, or put another way, $\overline{(x+y)}_n = \overline{x}_n + \overline{y}_n$. We must show that the mapping f above respects addition: $f(x+y) = f(x) + f(y)$, and we are looking at a direct product (some say *sum* here) of abelian groups in the target of f. So,

$$f(x+y) = (\overline{(x+y)}_{50}, \overline{(x+y)}_2) = (\overline{x}_{50}+\overline{y}_{50}, \overline{x}_2+\overline{y}_2) = (\overline{x}_{50}, \overline{x}_2)+(\overline{y}_{50}, \overline{y}_2) = f(x)+f(y).$$

b) Determine $\ker f$*.*

The kernel means (in additive notation)

$$\{x \in \mathbb{Z} \,|\, f(x) = 0\} = \{x \in \mathbb{Z} \,|\, \overline{x}_{50} = 0, \overline{x}_2 = 0\} = (50\mathbb{Z}) \cap (2\mathbb{Z}) = 50\mathbb{Z}.$$

c) Determine im $f = f(\mathbb{Z})$.

Evidently, the image consists of all pairs $(\overline{x}_{50}, \overline{x}_2)$ as x varies over all integers. Since $\mathbb{Z} \to \mathbb{Z}_2$ induces a mapping $g : \mathbb{Z}_{50} \to \mathbb{Z}_2$, we can use this to write $f(\mathbb{Z})$ to be the *graph* of g. See why this is correct.

d) Partition the following collection of groups into isomorphism classes:

$\mathbb{Z}_{100}$, $\mathbb{Z}_{50} \times \mathbb{Z}_2$, $\mathbb{Z}_{25} \times \mathbb{Z}_4$, $\mathbb{Z}_{20} \times \mathbb{Z}_5$.

All of the groups above have 100 elements. The student who remembers that $\mathbb{Z}_6 \simeq \mathbb{Z}_2 \times \mathbb{Z}_3$ but $\mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (and also *why*) should know what to look for here. We are given (in effect) four pairs of natural numbers: (100,1), (50,2), (25,4), and (20,5). In only the first and the third pair are the numbers relatively prime, and in both groups there is an element of order 100. Thus, the groups $\mathbb{Z}_{100}$ and $\mathbb{Z}_{25} \times \mathbb{Z}_4$ are cyclic, the others not. The highest order of an element of $\mathbb{Z}_{50} \times \mathbb{Z}_2$ is 50, and the highest order of an element of $\mathbb{Z}_{20} \times \mathbb{Z}_5$ is 20.

We see that $\mathbb{Z}_{100}$ and $\mathbb{Z}_{25} \times \mathbb{Z}_4$ are isomorphic, and there are no other isomorphisms.

[5,10] 4. a) Suppose that an element $g \neq 1$ in a group satisfies the equation $g^6 = 1$. Given that, what are the possibilities for the order of g?

The possibilities are 2,3,6 (1 is excluded by $g \neq 1$).

b) Determine $\{n \in \mathbb{N} \mid S_n \text{ contains an element of order } 6\}$.

It was all too easy to fall into the trap that the question was equivalent to determining when S_n contains a 6-cycle, which is possible exactly when $n \geq 6$. The trouble with that is that there is another way of getting a permutation of order 6, namely by taking the product of a 2-cycle with a disjoint (commuting) 3-cycle. That does occur in S_5. You should quickly check that in S_n with $n \leq 4$, every element has order at most 4. The answer is $\{n \in \mathbb{N} \mid n \geq 5\}$, affectionately written as $n \geq 5$.

[5,10,10,5] 5. S_6 is identified with the subgroup $\{\sigma \in S_7 \mid \sigma(7) = 7\}$ of S_7.

a) Calculate the index $[S_7 : S_6]$.

$[S_7 : S_6]$ denotes the number of S_6-cosets in S_7. Since S_7 is a finite group, the index is just $|S_7|/|S_6|$. This equals $7!/6! = 7$. People who wrote the difference instead of the ratio should repair the wiring immediately.

b) Show that S_6 is not a normal subgroup of S.

You know, it's kind of hard to make up a normal subgroup of permutations! The reason is that persistent formula saying explicitly what $\tau\sigma\tau^{-1}$ is. If you just apply this formula, you see how easy it is to get a cycle $\sigma \in S_6$ such that a conjugate $\tau\sigma\tau^{-1}$ is not in S_6. So one works with that formula in mind.

Lot's of choice! $(67)(123456)(67) = (123457) \notin S_6$. Yes, it's that simple, once you accept how permutations conjugate.

c) Let τ be the transposition $(6\ 7)$. Determine the right coset $S_6\tau$, and write it in the form $\{\sigma \in S_7 \mid \sigma(*) = \#\}$. Do not even think of listing the elements of S_6.

Yeah, who wants to list 720 elements? If you play around a little, you'll see that the answer is $\{\alpha \in S_7 \mid \alpha(6) = 7\}$. To explain this, let's first show that our coset is contained in the alleged answer. It's not difficult: if $\sigma \in S_6$, then for the coset element $\alpha = \sigma\tau$ we have $\alpha(6) = \sigma\tau(6) = \sigma(7) = 7$, and that's that.

Now how do we know that the alleged answer contains nothing else, i.e., just our coset? Well, if $\alpha(6) = 7$, we need to write $\alpha = \sigma\tau$, for some $\sigma \in S_6$ and τ as above. Said another way, we must see that $\alpha\tau \in S_6$. But that's just $\alpha\tau(7) = 7$, so $\alpha(6) = 7$. It is the same argument, run in reverse!

d) Show that there are no homomorphisms of S_7 onto S_6.

You are to know that homomorphisms have images and kernels, and the kernels are normal subgroups. Given what we said in part (b), ... The kernel would have to be of order 7. A normal subgroup of order 7, eh? It would have to be a cyclic group generated by a 7-cycle. The conjugates of that 7-cycle are other 7-cycles, by that persistent formula; indeed, all 7-cycles occur. We can get outside our little group.

[15] 6. Recall that a monoid is a set with one binary operation that obeys the associative law and has an identity element. Show that the set of units (elements with 2-sided inverses) in a monoid is a group (when one uses the multiplication of the monoid).

The purpose of this problem was to force you to show that you understood the axioms of a group. You had to show that the product of two units (in a monoid) is a unit, that the identity of the monoid is a unit, that the inverse of a unit is a unit, and that the associative law held. All of this comes from the monoid, of course, but I wanted to see you address the details.