Continuation of proof of the theorem on symmetric polynomials

We are going after the following:

Theorem. *Let F be a field with infinitely many elements. Then every symmetric polynomial in $(x_1, ..., x_n)$, with coefficients in F, is a polynomial over F in the elementary symmetric functions of these variables: $\sigma_1(x_1, ..., x_n)$, ..., $\sigma_n(x_1, ..., x_n)$.*

Said another way, one can always specify a homomorphism of rings:

$$\phi : F[x_1, ..., x_n] \to R,$$

where R is *any* commutative ring containing F, by just picking arbitrary values $\phi(x_j)$ $(1 \le j \le n)$, for ϕ is to respect multiplication. For $R = F[x_1, ..., x_n]$, we take $\phi(x_j) = \sigma_j(x_1, ..., x_n)$. Then ϕ maps *onto* the (sub-ring of) symmetric polynomials. Given what we did today:

Proof of theorem. The proof will go by induction on the number of variables n *and* the degree of the polynomial. Let $P(x_1, ..., x_n)$ be a symmetric polynomial. When we set $x_n = 0$, we get a symmetric polynomial in $n - 1$ variables

$$\widehat{P}(x_1, ..., x_{n-1}) = P(x_1, ..., x_{n-1}, 0).$$

By induction on n, $\widehat{P}$ can be written as a polynomial in the elementary symmetric functions in $n - 1$ variables. We notate this as

$$\widehat{P}(x_1, ..., x_{n-1}) = Q(\widehat{\sigma}_1, ..., \widehat{\sigma}_{n-1}).$$

We use this to say something about P itself; consider

$$(*) \qquad P(x_1, ..., x_n) - Q(\sigma_1, ..., \sigma_{n-1}),$$

where σ_1 means $\sigma_1(x_1, ..., x_n)$, etc. Well, (*) is a polynomial in $(x_1, ..., x_n)$ that vanishes (is zero) whenever $x_n = 0$. By what we said in class, x_n is a factor of (*). Now (*) *is* a symmetric polynomial of which x_n, hence every x_j, is a factor. It follows that (*) is divisible by $\sigma_n = \sigma_n(x_1, ..., x_n) = x_1 \cdots x_n$. We rewrite this as:

$$P(x_1, ..., x_n) = Q(\sigma_1, ..., \sigma_{n-1}) + \sigma_n R(x_1, ..., x_n).$$

where R is symmetric of lower degree than P. Take it from here.