Some solutions from HW4

I happened to notice how so many students fell into the trap. I'm talking about 2.3 #23: *Is every abelian subgroup of G contained in the center $Z(G)$ of G?*

The answer is "no". There are several ways of looking at this, so pick one. If you set out to prove that the answer is "yes", and write things down carefully, you will fail, suggesting that you start looking for a counterexample. It's actually quite easy to find one!

Let H be an abelian subgroup of G. One possibility for H is $Z(G)$. We have

$$Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

Note that H is abelian if and only if $Z(H) = H$, an assertion that does not involve the rest of G.

For any element $a \in G$, $H = \langle a \rangle$ is an abelian subgroup of G. If that were contained in $Z(G)$ for each $a \in G$, that would give $Z(G) = G$, i.e., that G is abelian.

So we can take *any* non-abelian group G, and a a non-central element. then $H = \langle a \rangle$ is an abelian group that is not contained in $Z(G)$. It's easy to specify such elements in S_3, also in the group of words on two generators, where the center is trivial.