

## Conjugates as friends

You should be picking up a vague feeling that conjugation in a [non-abelian] group represents a sort of change of point of view in the group. In the multiplicative group of invertible real  $n \times n$  matrices (or more generally, the multiplicative monoid of all real  $n \times n$  matrices), it displays the effect of a change of basis in  $\mathbb{R}^n$ , a well-known feature of Linear Algebra.

This notion has come up frequently in this course. Here are some things to keep in mind. Throughout, we are working in a group  $G$ :

1. For any  $a \in G$ , conjugation by  $a$  defines an automorphism of  $G$ . The formula for the automorphism is

$$\sigma_a(g) = aga^{-1}.$$

Note that  $\sigma_a(g) = g$  if and only if  $g$  and  $a$  commute ( $a$  centralizes  $g$ ). Thus,  $\sigma_a$  is the trivial automorphism if and only if  $a$  is in the center  $Z(G)$  of  $G$ ; it is trivial for *all*  $a$  if and only if  $G$  is abelian.

2. Conjugacy is an equivalence relation on the elements of  $G$ , namely  $g \sim h$  if and only if  $h = \sigma_a(g)$  for some  $a \in G$ . Note in particular that this gives  $g = \sigma_{a^{-1}}(h)$  (symmetry of the relation).

3.  $|\sigma_a(g)| = |g|$ . One might reason: *If  $g^n = 1$ , then  $(\sigma_a(g))^n = ag^n a^{-1} = 1$ , so  $|\sigma_a(g)| \leq |g|$ . (WHY?) By the symmetry noted in #2 above, the reverse inequality holds (i.e., switch the roles of  $g$  and  $\sigma_a(g)$ ).*

4. Note that in any group, if one takes two elements  $g$  and  $h$ , then  $gh$  and  $hg$  are conjugate:  $hg = h(gh)h^{-1}$ . By #3 above, they have the same order.

5. Let  $r$  and  $n$  be natural numbers, with  $r \leq n$ . Then all  $r$ -cycles in  $S_n$  are conjugate. We've seen this before:  $\tau(1 \dots r)\tau^{-1} = (k_1 \dots k_r)$  whenever  $\tau$  takes 1 to  $k_1, \dots, r$  to  $k_r$ . Given the  $k_i$ 's, use them to make a  $\tau$ . Since conjugacy is an equivalence relation, all being conjugate to  $(1 \dots r)$  means that all are conjugate.