A look on the Arzela-Ascoli theorem

The issue was to prove: *Let $\{f_n : I \to \mathbb{R}\}$ be a uniformly bounded, uniformly equicontinuous sequence of functions. Show the existence of a subsequence that converges uniformly on I.* We know from a little earlier that if there is such a limit, it will automatically be continuous.

We had looked at a countable dense subset $\{q_j\}$ of I, for which we had to do countably many maneuvers to secure convergence at just these points (this can be done *without* worrying about the continuity of the f_n's and the denseness of the q_j's. I want to do that **my way**; the argument is basically the same as the one in the book, but it is done with my own esthetic choices.

Start with showing: There is a subsequence $\{g_k\}$ of $\{f_n\}$ with the property that $\{g_k(q_j)\}$ (as a function of k) converges for each j, and moreover

(*) $$|g_k(q_j) - g_\ell(q_j)| < 1/j \quad \text{for all } k \text{ and } \ell.$$

How can we arrange this? Find any subsequence of f's for which $\{f(q_j)\}$ converges. Since this sequence is Cauchy, there are only finitely many k's and ℓ's for which (*) fails. Throw them away (don't worry, there are plenty more!). Set the first one aside; it shall be used as the first function in our final subsequence. Keep the rest of the sequence of f's. That is a subsequence of the original sequence.

Iterate on this: Having verified the condition (*) for $j < M$, we choose a subset of the sequence kept concerning q_{M-1}, so that it converges and satisfy (*) for $j = M$; it's always possible. Set aside the first element of that sequence of functions, and make it the M-th term of set of elements set aside.

This process generates a subsequence $\{g_k\}$ of $\{f_n\}$ that satisfies (*) (for **all** j), and also converges uniformly at the points of the sequence $\{q_j\}$ It is equivalent to say that the sequence of functions $\{g_k\}$ is uniformly Cauchy on $\{q_j\}$.

Take a Cauchy sequence of q_j's. (I won't change symbols.) For a *dense* sequence in I, we have for each $x \in I$ such a Cauchy sequence converging to x. (There are actually infinitely many.) We want to say something about $\{g_k(x)\}$. Call forth your triangles:

(**) $$d(g_k(x), g_\ell(x)) \le d(g_k(x), g_k(q_j)) + d(g_k(q_j), g_\ell(q_j)) + d(g_\ell(q_j), g_\ell(x)).$$

(I couldn't resist using metric space notation.)

At this point we have to loosen up a little. If you keep in mind that we might have started with a rearrangement of $\{q_j\}$, you'll see that there's nothing sacred about (*). Convince yourself that it suffices to answer the following: *Let $\varepsilon > 0$. For which j, k, ℓ can any one of the three terms in (**) be $\ge \varepsilon/3$?* (Explain.)