Cauchy sequences and the real numbers

We are really retreating into the past as we describe the construction of the real numbers $\mathbb{R}$ from the rationals numbers $\mathbb{Q}$. The underlying theme is that if we think we like the rational numbers, we might change our minds when we realize that there are sequences of rational numbers that look like they should have a limit, but don't (in $\mathbb{Q}$, of course.)

The main idea is to talk about **Cauchy sequences** (2.1). $\mathbb{R}$ is taken to mean the set equivalence classes of Cauchy sequences of rational numbers, under the **appropriate** equivalence relation.

1. A convergent sequence is Cauchy.

2. The sequence of truncations of an infinite decimal expansion, e.g.

$$\{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \dots\},$$

gives a Cauchy sequence of rational numbers.

3. Can we get by, using only infinite decimal expansions, as in #2 above? Even then, we must introduce an equivalence relation. For instance, we should be familiar with the notion that $1.\overline{0}$ and $0.\overline{9}$ (the latter's truncations are .9, .99, .999, ...) represent the same number, so they are declared to be equivalent infinite decimals.

4. Don't forget that we want to be able to see $\mathbb{R}$ come out as an ordered field. That can only come, somehow, from the fact that $\mathbb{Q}$ is an ordered field.