**Interior and boundary**

The *interior* of a subset $E$ of $\mathbb{R}$, often denoted $\text{int } E$, is defined to be the largest open set contained in $E$. For instance, the interior of $(a, b]$ is $(a, b)$. The meaning of *the largest open set contained in $E$*—is there such a thing?—can be given tautologically as the union of all open sets contained in $E$; here we use the property that the union of any collection of open sets is open. (On Feb 16, we defined the closure of $E$ as the smallest closed set containing $E$, where a similar issue arises.)

I think we have an idea of what is meant by the boundary of a country. What is a reasonable notion of the *boundary* of a set $E \subset \mathbb{R}$? It should give the expected outcome that the boundary of any interval with endpoints $a$ and $b$ consists of the two endpoints. Here it is: the boundary of $E$ is defined to be

$$b(E) = \overline{E} \cap \overline{E'},$$

where $E'$ is the complement of $E$, and the “bar” denotes closure.

**Exercises.**

1. a) Show that $b(E) = b(E')$.
   
   b) Show that $x \in b(E)$ if and only if every open interval that contains $x$, contains both a point of $E$ and a point of $E'$. [“a point” means “at least one point”]

2. a) Show that $\text{int } E$ and $b(E)$ are disjoint.
   
   b) Derive the formula $\overline{E} = (\text{int } E) \cup b(E)$.

3. Give a complete argument, in full detail, that shows that in any unbounded set of positive real numbers there is a sequence $\{x_j\}$ with

$$\lim_{j \to \infty} x_j = +\infty.$$

(I know it’s “obvious” ....)