Order of events

In today's lecture, I was going through the proof of the uniform continuity of a continuous function $f : D \to \mathbb{R}$, when D is compact. The order of events got confused.

We had two sequences of elements of D, called $\{x_{1,n}\}$ and $\{x_{2,n}\}$, with the property that for some fixed $\varepsilon > 0$,

$$(*) \qquad a) \ |x_{1,n} - x_{2,n}| < \tfrac{1}{n} \qquad \text{and} \qquad b) \ |f(x_{1,n}) - f(x_{2,n})| \geq \varepsilon$$

for all n. We were to show that this does not sit well with the compactness of D.

The correct order of events is:

1. Use the compactness of D: take a subsequence $\{x_{1,f(n)}\}$ of $\{x_{1,n}\}$ that converges. Call the limit y.
2. Take the corresponding subsequence $\{x_{2,f(n)}\}$ of $\{x_{2,n}\}$.
3. See that $\{x_{2,f(n)}\}$ also converges to y, using a) in (*).
4. Observe that the continuity of f at y is violated by b) in (*).

Let's work out #3, for that's where the trouble occurred.

$$\begin{aligned} |x_{2,f(n)} - y| &= |(x_{2,f(n)} - x_{1,f(n)}) + (x_{1,f(n)} - y)| \\ &\leq |x_{2,f(n)} - x_{1,f(n)}| + |x_{1,f(n)} - y| \qquad \text{(triangle inequality)} \end{aligned}$$

Both terms in the last line go to zero as $n \to \infty$..

Item #4 can be explained by: The definition of continuity of f at y gives

$$|f(x) - f(y)| < \frac{\varepsilon}{2}$$

for x sufficiently close to y, and that is inconsistent with the statement b) of (*) from our construction:

$$|f(x_{1,f(n)}) - f(x_{2,f(n)})| \geq \varepsilon \qquad \text{for all } n$$

(triangle inequality again).

I hope you can recognize that the ingredients were right in class, though the order of events was off.