

Problem #4: solution

a) The statement that $x_n \in (0, 3]$ for all n follows recursively (i.e., by induction), as follows: $x_1 = 2 \in (0, 3]$. If $x_n \in (0, 3]$,

$$(1) \quad x_{n+1} = 3/(1 + x_n),$$

which is clearly positive and less than 3.

b) Since the *closed* interval $[0, 3]$ is compact, there is a limit point in $[0, 3]$. To find a limit point in $(0, 3]$, one has to exclude the possibility that 0 is the only limit point of $\{x_n\}$ in $[0, 3]$, and that occurs if and only if $x_n \rightarrow 0$. This is rather easy: the assertion $x_n \in [1, 2]$ follows by the same reasoning as $x_n \in (0, 3]$. (Verify this.) Since $[1, 2]$ is compact, the sequence $\{x_n\}$ must have a limit point in $[1, 2]$, and that is contained in $(0, 3]$.

You may be sensing that we have used very little of the detail about $\{x_n\}$ in the solution to a) and b). At some point, like now if not sooner, we might calculate the numerical values of $x_1, x_2, x_3, x_4 \dots$. Compute: $x_1 = 2, x_2 = 1, x_3 = 3/2, x_4 = 6/5 \dots$. Go a little further and you should see evidence that the numbers go as follows: start, goes down, goes up by less, goes down by less than that, This would give the assertions of c). *Would give?! Yes, we need to verify that the apparent pattern is in fact correct, and that can only be deduced from (1).* [By the way, the existence of an increasing subsequence also implies b).]

c) The following works, though as with anything involving making estimates, other ways might work too. Use (1) in conjunction with taking differences:

$$(2) \quad x_{n+1} - x_n = \frac{3}{1 + x_n} - \frac{3}{1 + x_{n-1}} = -3 \frac{x_n - x_{n-1}}{(1 + x_n)(1 + x_{n-1})}.$$

This shows at once the “down, up, down, up, ...,” for that’s about the *sign* of the difference $d_{n+1} = x_{n+1} - x_n$. As for the “less and less”, it’s a question of the *size* (i.e., absolute value) of the differences. Taking absolute values in (2), we get

$$(3) \quad |d_{n+1}| = \frac{3|d_n|}{(1 + x_n)(1 + x_{n-1})} \leq \frac{3|d_n|}{4};$$

$|d_j|$ is a decreasing function of j .

d) Iterating (3) yields that

$$(4) \quad |d_n| \leq \left(\frac{3}{4}\right)^{n-2} |d_2| = \left(\frac{3}{4}\right)^{n-2}.$$

From (4), we use the triangle inequality to obtain:

$$(5) \quad |x_m - x_n| \leq \sum_{j=n}^{m-1} |d_j| \leq \sum_{j=n}^{m-1} \left(\frac{3}{4}\right)^{j-2} < 4\left(\frac{3}{4}\right)^{n-2}$$

whenever $m > n$. Here, we have invoked the fact that the geometric series with ratio $3/4$ converges. We see from (5) that $\{x_n\}$ is a Cauchy sequence, so it converges to a unique real number. There is thus only one limit point (namely the limit of the sequence; call it x). To evaluate this limit, note that it follows from (1) that $x = 3/(1+x)$, so $x^2 + x - 3 = 0$. We can solve this quadratic equation (and take, of course, the *positive* root).