

Breaking News

1. I have decided to hold the first midterm (Monday, March 2) in Shaffer 302 at the regular class time, padded a little, i.e., 1:25–2:50pm. I hope you don't mind.

2. The two usages of *limit point*, for sequences vs. subsets of \mathbb{R} , may cause some confusion, I concede, but I hope you can get it straight. Sometimes, one can find the term *accumulation point* used instead, but that too may occur in both cases. If I had to make a choice, I would use limit points for sequences and accumulation points for subsets. A minor headache!

3. I will be away on Monday Feb 23, and class is cancelled. However, I will write up what I would have covered in news items #5 and #6 below.

4. Submit your solutions to Assignment 5 through the mail slot in the door of Krieger 209 **by noon, Tuesday 2/24**. Be sure that the names of the course and of the TA (Longzhi Lin) are on the top page of your papers.

5. **More on compactness.** We have taken as our principal definition of compactness is: $E \subset \mathbb{R}$ is said to be compact when every sequence with terms in E has a limit point (in the sense of sequences) in E . This is how compactness often gets used. We had decided Wed 2/18 that a subset of \mathbb{R} is compact *only if* it is a closed and bounded subset. We were starting to prove the converse: a closed and bounded subset is compact. From that, we can use the simple fomulation of compact: closed and bounded.¹ After the simple observation that a closed subset of a compact set is compact (for an easy reason), we were reduced to showing:

Proposition. *Any closed interval is compact.*

Proof. Any two closed intervals are “the same” (under shifting and rescaling), it is enough to do it for $I = [0, 1]$. OK? Let $\{x_j\}$ be a sequence in I . We wish to find a limit point in I for the sequence; note that any limit point must be in I , since I is closed. But we have to do something!

Step 1. Partition I into two (closed) subintervals of equal length, i.e.,

$$I = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1].$$

At least one of the two must contain infinitely many terms x_j . Pick one that does, and call it I_1 , and pick one x_j that is in I_1 . Call that j “ $f(1)$ ”. We are going to iterate this procedure.

Step 2. Partition I_1 into two (closed) subintervals of equal length, At least one of the two must contain infinitely many terms x_j . Pick one that does, and call it I_2 , and pick one x_j with $j > f(1)$ that is in I_2 . Call that j “ $f(2)$ ”. Keep going

¹This actually holds in \mathbb{R}^n for any $n \in \mathbb{N}$, but not in infinite-dimensional spaces.

Let's describe the process. Given a closed subinterval I_n of length 2^{-n} , pick a half I_{n+1} a closed subinterval I_n of length 2^{-n-1} that contains infinitely many terms x_j . Pick $j > f(n)$ that is in I_{n+1} . Call that j " $f(n+1)$ ". In doing this, we are producing a subsequence of the given one. We have $I_{n+1} \subset I_n$, $x_{f(n)} \in I_n$, so for $m \geq n$, we have $x_{f(m)} \in I_n$ and

$$(*) \quad |x_{f(n)} - x_{f(m)}| \leq 2^{-n}.$$

$f : \mathbb{N} \rightarrow \mathbb{N}$ is by construction an increasing function, so the sequence $\{x_{f(j)}\}$ is a subsequence of $\{x_j\}$, and by (*) it is a It therefore has a limit in \mathbb{R} ,—call it y . Thus, y is a limit point of $\{x_j\}$. The limit must lie in I because

6. Even more on compactness. The most standard definition of *compact* is the one that says: E is compact when every open cover has a finite subcover (see the book). Roughly, if you use an infinite number of open sets (intervals) to cover a compact set, then you are being very wasteful! (Don't take that personally.)

I wanted to show you how one of those mysterious truths from Calc I, stated in the first lecture, follows directly from this covering property. This is not the *niciest* way to prove it, but (as usual) I'm not trying to be nice.

Proposition. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then f is bounded, i.e., there is $M \in \mathbb{R}^+$ such that $|f(x)| \leq M$ for all $x \in [a, b]$.*

Here's a sketch of the proof. Try to fill in the details. Write I for $[a, b]$. The hypothesis says that for every $y \in I$,

$$\lim_{x \rightarrow y} f(x) = f(y).$$

(We mean one-sided limits at the endpoints, *of course*.) It follows that y has a neighborhood U_y (one-sided at the endpoints²) on which f is bounded. Then $I = \bigcup_{y \in I} U_y$, an uncountable cover. But I is compact

²In order to avoid having to do a silly artificial construction across the endpoints, and to underscore the idea that we don't care how f might be extended to points outside of I , the following notion is both legitimate and useful. Let E be a subset of \mathbb{R} , and $U \subset E$. U is said to be *open relative to E* when $U = V \cap E$ for some open subset V of \mathbb{R} .