Completely dense

The title alludes to several points.

A. *The triangle inequality under manipulation.* It is easy enough to derive that

$$(1) \qquad |c - b| \geq |(|c| - |b|)|$$

for all $b, c \in \mathbb{Q}$, then all $b, c \in \mathbb{R}$: From $|a + b| \leq |a| + |b|$, we get

$$(2) \qquad |a| \geq |a + b| - |b|, \quad \text{so} \quad |c - b| \geq |c| - |b| \quad \text{(by substituting } c = a + b\text{)}.$$

This is not very informative when $|c| \leq |b|$. But switching the roles of b and c gives

$$(3) \qquad |b - c| = |c - b| \geq |b| - |c| = -(|c| - |b|),$$

and (2) and (3) together yield (1) — Calc 0.

B. *Let $x \in \mathbb{R}$. For every $n \in \mathbb{N}$, there exists $q \in \mathbb{Q}$ such that $|x - q| \leq 1/n$.* Indeed, we saw that if $\{q_j\}$ is any Cauchy sequence in $\mathbb{Q}$ representing x, $\lim_{j\to\infty} q_j = x$ in $\mathbb{R}$. By the definition of limit, we know that $|x - q_j| \leq 1/n$ when j is sufficiently large (as dictated by n). This says roughly that rational numbers are "seen everywhere" throughout $\mathbb{R}$. One says that $\mathbb{Q}$ is *dense* in $\mathbb{R}$.

C. *The real numbers are complete.* **This is the main point in introducing $\mathbb{R}$!** We talked of *denseness* in B; now for "completeness". That $\mathbb{R}$ is *complete* means that every Cauchy sequence in $\mathbb{R}$ is converges, i.e., has a limit in $\mathbb{R}$. (The analogous statement for $\mathbb{Q}$ is false, as we know.)

Given a Cauchy sequence $\{x_n\}$ of real numbers, represent each x_n by a Cauchy sequence $q_k(n)$ of rational numbers. Are we going to diagonalize again? Here's something to ponder: every Cauchy sequence in $\mathbb{Q}$ is equivalent to each of its *tails*, by which one means the sequences obtained by discarding the first N terms (for each N) of the given sequence. *Clear?* Replacing $q_k(n)$ by a suitable tail, we may assume that $|x_n - q_k(n)| \leq 1/n$ for all k. Continue, looking at $\{q_k(k)\}$ and showing that it's a Cauchy sequence in $\mathbb{Q}$ whose equivalence class is the limit of $\{x_n\}$. Or see the proof of completeness in the book (essentially the same), which uses the denseness from B. Be sure to note the role of the triangle inequality.