Completely dense

The title alludes to several points.

A. *The triangle inequality under manipulation.* It is easy enough to derive that

\[(1) \quad |c - b| \geq |(|c| - |b|)|\]

for all \(b, c \in \mathbb{Q}\), then all \(b, c \in \mathbb{R}\). From \(|a + b| \leq |a| + |b|\), we get

\[(2) \quad |a| \geq |a + b| - |b|, \quad \text{so} \quad |c - b| \geq |c| - |b| \quad (\text{by substituting } c = a + b).\]

This is not very informative when \(|c| \leq |b|\). But switching the roles of \(b\) and \(c\) gives

\[(3) \quad |b - c| = |c - b| \geq |b| - |c| = -(|c| - |b|),\]

and (2) and (3) together yield (1) — Calc 0.

B. Let \(x \in \mathbb{R}\). For every \(n \in \mathbb{N}\), there exists \(q \in \mathbb{Q}\) such that \(|x - q| \leq 1/n\). Indeed, we saw that if \(\{q_j\}\) is any Cauchy sequence in \(\mathbb{Q}\) representing \(x\), \(\lim_{j \to \infty} q_j = x\) in \(\mathbb{R}\). By the definition of limit, we know that \(|x - q_j| \leq 1/n\) when \(j\) is sufficiently large (as dictated by \(n\)). This says roughly that rational numbers are “seen everywhere” throughout \(\mathbb{R}\). One says that \(\mathbb{Q}\) is dense in \(\mathbb{R}\).

C. The real numbers are complete. This is the main point in introducing \(\mathbb{R}\)! We talked of denseness in B; now for “completeness”. That \(\mathbb{R}\) is complete means that every Cauchy sequence in \(\mathbb{R}\) is converges, i.e., has a limit in \(\mathbb{R}\). (The analogous statement for \(\mathbb{Q}\) is false, as we know.)

Given a Cauchy sequence \(\{x_n\}\) of real numbers, represent each \(x_n\) by a Cauchy sequence \(q_k(n)\) of rational numbers. Are we going to diagonalize again? Here’s something to ponder: every Cauchy sequence in \(\mathbb{Q}\) is equivalent to each of its tails, by which one means the sequences obtained by discarding the first \(N\) terms (for each \(N\)) of the given sequence. Clear? Replacing \(q_k(n)\) by a suitable tail, we may assume that \(|x_n - q_k(n)| \leq 1/n\) for all \(k\). Continue, looking at \(\{q_k(k)\}\) and showing that it’s a Cauchy sequence in \(\mathbb{Q}\) whose equivalence class is the limit of \(\{x_n\}\). Or see the proof of completeness in the book (essentially the same), which uses the denseness from B. Be sure to note the role of the triangle inequality.

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