1. **It happens.** We considered two functions $f : [0, 1] \to \mathbb{R}$:

\[
f_1(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q} \\
0 & \text{otherwise}
\end{cases}
\]

\[
f_2(x) = \begin{cases} 
1/q & \text{if } x \in \mathbb{Q}, \ x = p/q \text{ in lowest terms} \\
0 & \text{otherwise}
\end{cases}
\]

Carry out the formal argument (with $\varepsilon$, $\delta$ or equivalent) to show that $f_1$ is discontinuous everywhere on its domain, and that $f_2$ is continuous precisely on the irrational numbers in $[0, 1]$.

2. **At the bottom.** There are two ways to look at the second derivative test of a critical point $x_0$ for a local extremum (so in the interior of its domain) of a twice-differentiable function. We’ll do the case of a local minimum (stand on your head to do a local maximum). Thus, $f'(x_0) = 0$ and $f''(x_0) > 0$:

   a) **By Taylor’s Theorem with Lagrange form of the Remainder.** Regrettably, this requires $f$ to be $C^2$ (i.e., $f$ has a continuous second derivative). We have:

\[
f(x) = P_1(x) + R_2(x) = f(x_0) + f'(x_0)(x - x_0) + f''(c)(x - x_0)^2/2,
\]

for some $c$ between $x$ and $x_0$. With $f'(x_0) = 0$, this reads

\[
f(x) = f(x_0) + f''(c)(x - x_0)^2/2
\]

With $x$ close to $x_0$, $c$ must be even closer to $x_0$; with $f$ $C^2$, when $x$ is sufficiently close to $x_0$, $f''(c)$ must also be positive, so $f(x)$ is greater than $f(x_0)$ by (1).

   b) **By basic principles concerning the derivative.** We will not assume $f$ to be $C^2$; all we need here is the existence of $f''(x_0)$ (doesn’t that make you feel strong?). When $f''(x_0) > 0$, that means, of course that $g = f'$ has positive derivative at $x_0$. Use the Mean Value Theorem (Taylor’s Theorem for $P_0$):

\[
f(x) = f(x_0) + f'(c)(x - x_0),
\]

for some $c$ between $x$ and $x_0$.

Go back to the definition of derivative, and look at the difference quotient in the definition of the derivative:

\[
\frac{g(x) - g(x_0)}{x - x_0} = \frac{f'(x) - f'(x_0)}{x - x_0}.
\]

As $x \to x_0$, the quantity (3) becomes positive, by assumption. With $f'(x_0) = 0$, it follows that $f'(c) > 0$ when $c$ is a little greater than $x_0$, and $f'(c) < 0$ when $c$ is a little less than $x_0$. Use this information in conjunction with (2) to see the local minimum.