

From 3/23

1. *It happens.* We considered two functions $f : [0, 1] \rightarrow \mathbb{R}$:

$$f_1(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(x) = \begin{cases} 1/q & \text{if } x \in \mathbb{Q}, x = p/q \text{ in lowest terms} \\ 0 & \text{otherwise} \end{cases}$$

Carry out the formal argument (with ε , δ or equivalent) to show that f_1 is discontinuous everywhere on its domain, and that f_2 is continuous precisely on the irrational numbers in $[0, 1]$.

2. *At the bottom.* There are two ways to look at the second derivative test of a critical point x_0 for a local extremum (so in the interior of its domain) of a twice-differentiable function. We'll do the case of a local minimum (stand on your head to do a local maximum). Thus, $f'(x_0) = 0$ and $f^{(2)}(x_0) > 0$:

a) *By Taylor's Theorem with Lagrange form of the Remainder.* Regrettably, this requires f to be C^2 (i.e., f has a continuous second derivative). We have:

$$f(x) = P_1(x) + R_2(x) = f(x_0) + f'(x_0)(x - x_0) + f^{(2)}(c)(x - x_0)^2/2,$$

for some c between x and x_0 . With $f'(x_0) = 0$, this reads

$$(1) \quad f(x) = f(x_0) + f^{(2)}(c)(x - x_0)^2/2$$

With x close to x_0 , c must be even closer to x_0 ; with f C^2 , when x is sufficiently close to x_0 , $f^{(2)}(c)$ must also be positive, so $f(x)$ is greater than $f(x_0)$ by (1).

b) *By basic principles concerning the derivative.* We will not assume f to be C^2 ; all we need here is the existence of $f^{(2)}(x_0)$ (doesn't that make you feel strong?). When $f^{(2)}(x_0) > 0$, that means, of course that $g = f'$ has positive derivative at x_0 . Use the Mean Value Theorem (Taylor's Theorem for P_0):

$$(2) \quad f(x) = f(x_0) + f'(c)(x - x_0),$$

for some c between x and x_0 .

Go back to the definition of derivative, and look at the difference quotient in the definition of the derivative:

$$(3) \quad \frac{g(x) - g(x_0)}{x - x_0} = \frac{f'(x) - f'(x_0)}{x - x_0}.$$

As $x \rightarrow x_0$, the quantity (3) becomes positive, by assumption. With $f'(x_0) = 0$, it follows that $f'(c) > 0$ when c is a little greater than x_0 , and $f'(c) < 0$ when c is a little less than x_0 . Use this information in conjunction with (2) to see the local minimum.