Rearranging

In lecture on April 6, I was talking about rearrangements of a conditionally convergent series. We talk about it further here.

It should be clear that the determination of limit points for the sequence of partial sums was purely abstract, independent of the actual terms of that sequence. One needs to use only the following:

For a conditionally convergent series, one separates the positive and negative terms of the series; there must be infinitely many of each, and both the series of positive terms and the series of negative terms must diverge.

This implies that for both subsequences, the terms go to zero, but there exist segments of both, starting arbitrarily far out, that are arbitrarily large. (That's why one has to suffer through convergence testing!)

At issue was showing that a conditionally convergent series can have two limit points; call them a and b. It seemed that every number between a and b had to be limit points as well. Let's prove that:

Proposition. *Let a and b be limit points of a rearrangement of a conditionally convergent series, with $a < b$. Here we can allow the extended reals $a = -\infty$ or $b = \infty$ (or both). Then every point $c \in (a, b)$ is also a limit point.*

Proof. Again, we think of graphing s_n as a function of n in the usual (x, y)-plane. Then connect, for each n, the points (n, s_n) and $(n + 1, s_{n+1})$ by a line segment, as in early childhood, to make the graph of a continuous function on $[1, \infty)$. The key observation is: for any $c \in (a, b)$, the graph must cross $y = c$ infinitely often. Let $\varepsilon > 0$. Since $a_n \to 0$, one has $|a_n| < \varepsilon$ when n is sufficiently large. Suppose that $s_n < c$ but $s_{n+1} \geq c$ (or vice versa: $s_n \geq c$ but $s_{n+1} < c$). For such n, with n sufficiently large in the above sense,

$$|s_n - c| \leq |s_n - s_{n+1}| = |a_{n+1}| < \varepsilon.$$

Since the above holds for all $\epsilon > 0$, c is a limit point of $\{s_n\}$.