110.202 Calculus III Professor Zucker
Final Exam: December 13, 2002
Time allowed: 3 hours

No books, no notes, no calculators or other devices! Write legibly, and show all relevant work—or risk losing credit. Your solutions will be judged as answers to what is asked.

1. Let $C$ be the curve with parametric equations: $x = t^3$, $y = t^5$, $z = t^7$, $t \in [1, 2]$. Evaluate $\int_C x^2 z \, dy$.

2. Let $\hat{C}$ be the upper half of $C$. Evaluate $\int_{\hat{C}} x^5 \, dx + y^2 \, dy$.

2 (cont’d). b) Let $\hat{C}$ be the upper half of $C$. Evaluate $\int_{\hat{C}} x^5 \, dx + y^2 \, dy$.

[30] 2. Let $C$ be the circle with polar equation $r = \cos \theta$, taken in the counterclockwise sense.

a) Use Green’s Theorem to evaluate $\int_C y \, dx$. 


3. a) Complete the definition of the partial derivative (in two variables) with respect to \( y \):
\[
\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}
\]
b) Give an example of a function \( f(x, y) \), defined for all \((x, y)\), for which \( f_x(0, 0) = f_y(0, 0) = 0 \) yet \( f \) is not differentiable at the origin.

4. For this problem, let \( C \) be the unit circle in the \((x, y)\)-plane, and \( D \) be the closed unit disc. (Thus, the boundary of \( D \) is \( C \).)

a) Let \( f(x, y) = \cos^2(\pi x) \sin(\pi y) \). Compute the value of \( f_y \) at an interior point of \( D \) of your choice (you must choose one).

b) Let \( g(x, y) = x^2 y \). Note that the point \( R = \left( \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right) \) lies on \( C \). Determine the derivative of \( g \) at the point \( R \) in the direction tangent to \( C \) (with its usual counterclockwise orientation).
5. Determine the maximum and minimum values of the function $x^3 + y^3$ on the elliptical region

$$x^2 + \frac{y^2}{4} \leq 1.$$
Let \( S \) be the surface given parametrically by \( x = \sin u \cos v, \ y = \sin u \sin v, \ z = \cos u \), with \( u \in [-\frac{\pi}{2}, \frac{\pi}{2}] \) and \( v \in [0, \pi] \).

a) Give a function \( f(x, y, z) \)—not identically zero!—such that \( S \) is contained in \( \{(x, y, z) : f(x, y, z) = 0\} \).

b) With \( f \) as in a), determine whether \( S = \{(x, y, z) : f(x, y, z) = 0\} \).

c) Compute the fundamental vector product for this parametrization.

[10] d) Take as given that the formulas provide a parametrization of \( S \) that is effectively one-to-one. Calculate \( \iint_S z^2 \, d\sigma \).
[40] 8. Let $S$ be the surface obtained by revolving about the $z$-axis (in space) the curve $C$

(*) \[ y = z^8 \sin\left(\frac{\pi z^2}{4}\right), \quad z \in [1, 2] \]

in the $(y, z)$-plane.

[10] a) Determine the boundary of $S$.

[10] c) Determine a nowhere-zero normal field to $C$ in the $(y, z)$-plane.

[15] d) Determine the flux of $\nabla \times (yi)$ out of $S$ in the normal direction to $S$ given by projecting the vector field $r$ (along $S$).

[5] b) Calculate $\nabla \times (yi)$. 
9. Let $E$ be the surface with parametric equations $x = (2 + \sin u) \cos v$, $y = (2 + \sin u) \sin v$, $z = \cos u$, with $u \in [0, 2\pi]$ and $v \in [0, 2\pi]$.

a) Calculate the area of $E$.

b) Give an example of a common object whose shape is like $E$. 

110.202 Calculus III  
Name: ____________________