Tangents and normals

1. By thinking of components, a vector that’s a function of \( t \) is the same as three scalar functions of \( t \) taken together. When \( \mathbf{r}(t) \) is a given vector function, we have the convention of viewing it as tracing a curve in space. The derivative \( \mathbf{r}'(t) \) is, of course, another vector function, corresponding to the derivatives of the three scalar functions, and it gives tangent vectors along that curve. Here, though, it does not help to think of \( \mathbf{r}'(t) \) as tracing another curve. If we want all of our tangent vectors to be the same length, we can form the unit tangent vector in the direction of \( \mathbf{r}'(t) \):

\[
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||},
\]

defined for all \( t \) such that \( \mathbf{r}'(t) \neq \mathbf{0} \).

If, for example \( \mathbf{r}(t) = ti - t^2j + t^3k \), then \( \mathbf{r}'(t) = i - 2tj + 3t^2k \), so

\[
\mathbf{T}(t) = \frac{i - 2tj + 3t^2k}{\sqrt{1 + 4t^2 + 9t^4}} = Q(t)i - 2tQ(t)j + 3t^2Q(t)k,
\]

where

\[
Q(t) = (1 + 4t^2 + 9t^4)^{-\frac{1}{2}}.
\]

2. There is no notion of the normal direction to a curve \( \mathbf{r}(t) \) at some \( t = t_0 \). However, the normal plane can be defined to be the plane through \( \mathbf{r}(t_0) \) that has \( \mathbf{r}'(t_0) \) as a normal vector (i.e., is perpendicular to the tangent line), provided of course \( \mathbf{r}'(t_0) \neq \mathbf{0} \). From the information here, we plug into the formulas from 12.6 to get the equation of the normal plane at \( t_0 \):

\[
(\mathbf{r} - \mathbf{r}(t_0)) \cdot \mathbf{r}'(t_0) = 0.
\]

Writing \( a, b \) and \( c \) for the components of \( \mathbf{r}'(t_0) \), we get for the equation of the plane:

\[
a(x - x(t_0)) + b(y - y(t_0)) + c(z - z(t_0)) = 0.
\]

For instance, the equation of the normal plane to \( ti - t^2j + t^3k \) at \( t = 1 \) is .... Do the calculations: \( \mathbf{r}(1) = i - j + k; \mathbf{r}'(1) = i - 2j + 3k \). Therefore, the equation of the normal plane is:

\[
(x - 1) - 2(y + 1) + 3(z - 1) = 0.
\]