On Green's Theorem

I don't want to overload you, but the course cannot stop just because we had an exam, nor because we will be having turkey. We have entered the heart of the course, not the tail! I just want to point out two things:

1. If I wanted to tell you what the advantages of having Green's theorem are, I would say: *It gives us a way of computing line integrals over closed curves—under conditions—without ever touching antiderivatives or parametrization.* This is accomplished by equating the given line integral to a double integral, or even to the line integral over some other closed curve that is easier to deal with.

2. I asked you to consider why Green's Theorem is true on the unit square,

$$\Omega = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\} = I \times I,$$

where I denotes the interval [0,1]. The boundary C of Ω is a piecewise smooth curve, whose segments are:

$$C_1\colon\ x = 1,\ y \in I; \quad C_2\colon\ y = 1,\ x \in I; \quad C_3\colon\ x = 0,\ y \in I, \quad C_4\colon\ y = 0,\ x \in I.$$

Since C is oriented counterclockwise in Green's Theorem, we see that C_2 and C_3 are taken in the opposite direction from which one usually tranverses I, while C_1 and C_4 go the "usual" way.

You were asked to note:

$$\begin{aligned}\iint_\Omega (Q_x - P_y)dA &= \int_0^1 \int_0^1 (Q_x(x,y) - P_y(x,y))dxdy \\ &= \int_0^1 \int_0^1 Q_x(x,y)dxdy - [\text{the } P_y \text{ integral}].\end{aligned}$$

We will check that $\int_0^1 \int_0^1 Q_x(x,y)dxdy$ is equal to $\oint_C Qdy$. By the fundamental theorem in one variable (y is being fixed),

$$\int_0^1 Q_x(x,y)dx = Q(1,y) - Q(0,y).$$

For this, Q_x is being considered inside the curve C, thus the condition. Therefore,

$$\begin{aligned}\int_0^1 \int_0^1 Q_x(x,y)dxdy &= \int_0^1 (Q(1,y) - Q(0,y))dy \\ &= \int_0^1 (Q(1,y)dy - \int_0^1 (Q(0,y)dy \\ &= \int_{C_1} Qdy + \int_{C_3} Qdy.\end{aligned}$$

This equals $\oint_C Qdy$: though C_2 and C_4 seem to be missing, we note that $dy = 0$ on these two sides, so they don't contribute anything to the integral over C.

One obtains Green's theorem from that, and the analogous calculation:

$$\int_0^1 \int_0^1 P_y(x,y)dxdy = -\oint_C Pdx.$$