What's really new in integration in Calculus III?

I decided to take a look into the question. The attentive student knows that all integrals are defined in terms of Riemann sums (5.2, 16.2, etc.), **but who wants to deal with those?!** Here's what I came up with:

I. Integration over curves.

a) *On the line.* The FTC (5.4) relieves us of the burden of dealing with integrals as they were defined. *This is Calc I, and a touch of Calc II.* This is pretty abstract stuff, since we all know we can't actually find an explicit formula for the antiderivative for the general function. Do you agree?

b) *In the plane and in space.* This involves parametrization of curves—one independent variable generically denoted t. It then tells us how to compute arclength via (a) on the t-line. *This is Calc II, and a touch of Calc III.* Regrettably, the integrations involved in the calculation of arclength usually cannot be worked out explicitly. But that is OK, for it didn't bother us in (a).

Who balks at the notion of the length along a curve? Any curve can be parametrized by arclength; the curve is viewed as a thin tape measure. As in (a), it again shouldn't bother us that it usually gives no explicit formula.

c) We barely get into Calc III by upping the stakes from (b) above to integrating *functions* with respect to arclength (17.4.1).

d) Then come the line integrals; what an awful name! For now, let's call them *oriented curve integrals.* This is real Calc III. Given an oriented curve C, and a differential $\omega = P(x,y)dx + Q(x,y)dy$, we need to know what one means by

$$\int_C \omega,$$

or at least how to go about calculating them. What needs to be said now? Isn't the point that on a curve, x and y are not independent? so dx and dy aren't either. Write everything in terms of t (for a parametrization that gives the correct orientation), and then get an integration as in (a). So what if we are unable to do the integration; it didn't bother us before! At least, there are only two possible orientations for a curve (and for a surface too later on), and changing the orientation simple changes the sign of the integral. That's why we write C and $-C$. You can compute with the wrong orientation, then change the sign if you like!

e) In the setting of (d), breathe a sigh of relief in the case where $\omega = dg$ for some function g defined in a neighborhood of C. This is saying we've cracked the anti-derivative barrier, and are in the analogue of the good part of (a), since we have a version of the FTC in the present situation (17.2). Know when to use it!

That's about it for curves.

II. Integration over surfaces.

a) *In the plane.* The double integral of a function over a plane region is accessible Calc III material. If we had inequalities in x and y that define the region, then we could try to calculate the double integral by repeated (iterated) integration. If the explicit integration is not possible—we should be used to that by now (see I).

b) Green's Theorem for regions with boundary curves (17.5) can most quickly be stated as: Let $\partial\Omega$ denote the *oriented* boundary of the plane region Ω. Then, with notation as in I(d),

$$\int_{\partial\Omega} \omega = \int_{\Omega} \ldots\ldots\ldots\, dA$$

where $Q_x - P_y$ goes in the blank —under conditions (get them straight!). The orientation of the outer boundary curve is counterclockwise, that of the inner boundary curves (if any) is clockwise. If the conditions are satisfied, you can use Green's Theorem to convert (say) a line integral over a simple closed curve to a double integral over the region it bounds. If I heard you grumble that I(d) makes you nauseous, then what I just said should make you beam with delight!

c) *In space.* This involves parametrization of surfaces (17.6), or pieces thereof—two independent variables generically denoted (u, v). It is rather difficult to explicitly parametrize surfaces except in standard cases: graphs $z = f(x, y)$, ellipsoids (incl. spheres), cylinders over curves, a torus (surface of a doughnut)—what else? It then tells us how to compute surface area via (a) above in the (u, v)-plane. *This is Calc III all right!* Regrettably, the integrations involved in the calculation of surface area usually cannot be worked out explicitly. ... (This again!)

d) We barely go deeper into Calc III by upping the stakes from (c) above to integrating *functions* with respect to surface area (17.7).

e) Then come the oriented surface integrals: the flux of a vector field. The orientation of a surface can be described as a choice of unit normal field along the surface, which tells you which way we are taking as the direction through the surface. The orientation is explicitly in the integrand that defines the flux. There are two orientations for a surface: one, and its negative.

f) Stokes' Theorem is Green's Theorem generalized to surfaces that are not planar. It says: Let S be a surface with boundary curves, and let ∂S denote the oriented boundary of S. Let $\mathbf{v}$ be a sufficiently differentiable vector field defined on S. Then

$$\int_{\partial S} \mathbf{v} \bullet dr = \iint_S (\nabla \times \mathbf{v}) \bullet \mathbf{n}\, d\sigma.$$

Did you grumble that II(c) makes you nauseous? Then what I just said should make you happy!

III. Integrals over solids.

a) We talk only about solids in space. (Going higher *is* possible, though we can't directly see that.) This is the triple integral of (16.9), which is basically of the same nature as the double integral, just more complicated. It is subject to the same sort of computational methods (repeated integration and change of variables formula).

b) The Gauss Divergence Theorem is a "surefire" way to convert a surface integral into a triple integral, provided it is a *flux integral* over a *closed surface (i.e., without boundary).* The statement is Theorem 17.9.2. It's the analogue of Green's Theorem one dimension higher. Did you grumble that II(c) makes you nauseous? Then what I just said should make you beam with delight!

Leftovers. From each lecture on Wednesday, I omitted doing something, and it was a different thing for each class. Here are the omitted things.

Lecture 1. A second parametrization of the upper half of the unit sphere comes from the fact that it is the graph of a function:

$$x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0 \quad \Leftrightarrow \quad (x, y) \in D \text{ and } z = \sqrt{1 - x^2 - y^2},$$

where D is the unit disc in the (x, y)-plane. (This is exactly parallel to the upper half of the unit *circle* in the plane; see it in the (y, z)-plane by putting $x = 0$ in the preceding).

The general graph $z = f(x, y)$, with $(x, y) \in \Omega$, is parametrized by the tedious but easy: $x = u$, $y = v$, $z = f(u, v)$, $(u, v) \in \Omega$. (In other words, *a graph comes with a parametrization built in!*) There is a natural one-to-one correspondence between curves in Ω and curves on the graph: To the curve $x = x(t)$, $y = y(t)$ in Ω, add a third expression for the graph: $z = f(x(t), y(t))$. We do that instead of taking $z = 0$, which would leave you in the (x, y)-plane.

Lecture 2. A flux question, followed by use of the Divergence Theorem, led to the integral.

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x + y + z) dz dy dx.$$

The region of integration is the solid bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$. Yes, it's a routine integration, but I spoke of the urge to change variables, making $x + y + z$ one of them .... What do you think?