On the material of Week 7

I'm feeling a little estranged from you people by my hospitalization, surgery and recovery period. I want to make some basic conceptual points about the material of Week 7. It is possible that you understand them already, but I offer them nonetheless.

1. a) A point of view on differentiability in two or more variables is that the derivative of a function at a point is a **vector**, and that vector is ∇f. Having a derivative is about *linear approximation* (page 862 of the textbook), something that is automatic in one variable, but not so easy in two or more. In practice, ...

b) Let $\mathbf{v}$ be a *non-zero* vector, which is taken to be fixed but arbitrary in this paragraph. Let $\mathbf{u}$ be a unit vector, which we treat as a variable. Then the dot product $\mathbf{v} \bullet \mathbf{u}$ takes on values between $-\|\mathbf{v}\|$ and $\|\mathbf{v}\|$. Indeed, $\mathbf{v} \bullet \mathbf{u} = \|\mathbf{v}\| \cos\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{u}$ (see (12.4.7)), since $\|\mathbf{u}\| = 1$. This number is maximized to be $\|\mathbf{v}\|$ when $\theta = 0$, where $\mathbf{u}$ is pointing in the direction of $\mathbf{v}$, and is minimized when $\theta = \pi$, where $\mathbf{u}$ is pointing in the direction opposite to $\mathbf{v}$. This is what is behind (15.2.7).

c) A rough attitude on the Chain Rule (in its incarnation as Theorem 15.3.4) is that, for matters of first-order approximation of a function differentiable at a point, a curve passing through a point can be replaced by its tangent line there. The latter need not be traced out with unit speed, but rather with the speed dictated by how the curve is parametrized (i.e., how fast one is tracing the curve at the point).

2. Something we have heard about this week concerns normal directions to level surfaces of differentiable functions of three variables:

$$f(x, y, z) = c,$$

the locus where f is constant with value c. An easy example of a level surface is a sphere or other quadric, but we have seen enough to know that level surfaces can look rather lousy, with corners, kinks, and even worse. Certainly, we have seen level *curves* (for functions of two variables) that made us say something other than "that's a smooth curve", and it's unreasonable to expect matters to get better when one passes from two variables to three. We have from (15.4.1):

Theorem. *If $\nabla f(x, y, z) \neq \mathbf{0}$, and $f(x, y, z) = c$, then $\nabla f(x, y, z)$ points in the direction normal to the level surface with equation $f(x, y, z) = c$.*

This statement is simple enough, but it might leave some of you a little cold. We should have a good sense of the normal direction for a plane (level surface of a linear function). Then general theory should and does tell us that when $\nabla f(x, y, z) \neq \mathbf{0}$, in a sufficiently small neighborhood of the point (x, y, z) above, the picture of the level surface is "like" the same for a plane.

It may help to get started by considering the issues one dimension lower of smoothness for curves. An old *Calc II* file on that topic is item #5 below.

3. When you have the problem of finding extreme values on a region (say, in the plane) with smooth boundary, it really goes as in Calc I: look for interior critical points, then check the boundary (language as in 14.5). However, the boundary is no longer just a pair of points. You have two options for dealing with the boundary:

either parametrize it and do a Calc I problem to find extrema for the restriction of the function to the boundary, or use Lagrange multipliers. If the boundary is not smooth or has infinite length, you may have to be disciplined or a little clever.

4. When using the Lagrange multiplier method, don't forget that being on the level set is part of the data; don't drop the equation $g(\mathbf{x}) = 0$ (the side condition) from your collection of information. Dropping the side condition will sabotage your solution attempt.

5. As I said in #2, it may help to go through the notion of smoothness of curves again. The following is a mildly edited version of an old Calc II file I wrote on the topic.

Velocity and smoothness of curves. Recall that a parametrized curve is given by equations of the form

$$x = x(t), \quad y = y(t), \quad \text{for } t \text{ in some interval } I. \tag{1}$$

We may always regard t as time, unless there is a good reason not to. Then, the equations describe the curve traced out by a particle moving in the plane. For any particular moment of time $t_0 \in I$, let $x_0 = x(t_0)$, $y_0 = y(t_0)$. This just says that our notation for where the particle is when $t = t_0$ is (x_0, y_0). It is *somewhere.* (If one doesn't establish sufficient notation for the mathematical situation at hand, one often ends up wishing later on that one had, and you may end up having to start over.)

Now, we understand further that $x(t)$ and $y(t)$ in (1) are to be **continuously differentiable** functions of t. This is the mathematically correct way of saying that we are talking only about situations in which it makes sense say how fast x and y are changing at any time in our t-interval I, and moreover that these rates of change vary continuously with time. In practice, ...

We get to the main point (you may wish to consult Section 7.1 of the textbook). First, forget about y. **Then we're back to Calculus I in the (t, x)-plane:** x is graphed as a function of t. If we know (say) that $x'(t_0) > 0$, by continuity the same inequality holds for all t in some small enough interval—call it I_0 – about t_0; one often says: *for t sufficiently close to t_0.* (What constitutes "sufficiently close" depends on the curve). It follows that $x(t)$ is an increasing function for $t \in I_0$. Therefore, the restriction of $x(t)$ to $t \in I_0$ has an inverse, which we write as

$$t = T(x),$$

where T is, like $x(t)$, continuously differentiable (near x_0), and

$$T'(x) = 1/x'(t), \text{ when } x = x(t) \ (t = T(x))$$

(see Theorem 7.1.8 of the text).

It's time to remember $y(t)$. For $t \in I_0$, we have that

$$y = y(t) = y(T(x)),$$

from which follows what I'm after:

a) the portion of the curve in question ($t \in I_0$) is the graph of y as a nice (i.e., differentiable) function of x;

b) The Chain Rule gives that $dy/dx = y'(T(x))T'(x)$, which equals $y'(t)/x'(t)$.

We can give an analogous discussion if $x'(t_0) < 0$, for then x is a *decreasing* function of t near t_0, and we continue as above. However, you should recognize that **there is no general statement that one can deduce from the remaining possibility:** $x'(t_0) = 0$.

To summarize: When $x'(t_0)$ is not 0 then for t sufficiently near t_0 the parametric equations in (1) actually define y as a differentiable function of x, and formula (b) above holds.

As far as the parametric equations go, there is no reason to prefer one coordinate in the plane over the other. Thus, we have the parallel assertion with the roles of x and y interchanged: **When $y'(t_0)$ is not 0, then for t sufficiently near t_0 the parametric equations in (1) actually define x as a differentiable function of y, and $dx/dy = x'(t)/y'(t)$.**

Next, note that both $x'(t_0) = 0$ and $y'(t_0) = 0$ *if and only if* $[x'(t_0)]^2 + [y'(t_0)]^2 = 0$. The above can be rephrased as: **At any instant of time t_0, if $x'(t_0)^2 + y'(t_0)^2$ is not 0, then the portion of the curve traced, for t sufficiently near t_0, is a graph of one of the variables x and y as a continuously differentiable function of the other.** (This may remind you a little of the theory of implicit differentiation (Section 3.7 in the textbook)).

For those of you who know about vectors (or want to know), we can make a vector out of the rates of change:

$$\mathbf{v}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}.$$

The condition above is equivalent to saying that the *length* of $\mathbf{v}$ at $t = t_0$, also known as the *speed* of the particle at that instant, satisfies:

$$\|\mathbf{v}(t_0)\|^2 = [x'(t_0)^2 + y'(t_0)^2] \neq 0.$$

The conclusion can be restated informally as: if the particle is moving according to nice functions of time, it is tracing a smooth curve whenever it is not instantaneously standing still. If the latter holds at t_0, the velocity vector $\mathbf{v}(t_0)$ is then tangent to the curve at (x_0, y_0).

If we happen to see how to explicitly eliminate the parameter from (1), i.e., write the local inverse functions explicitly, then we can see all of the above conclusions explicitly as well. For instance, consider the usual parametrization of the unit circle:

$$x = \cos t, \quad y = \sin t,$$

and take $t_0 = 0$, so $(x_0, y_0) = (1, 0)$. We then have $x'(0) = -\sin 0 = 0$, but $y'(0) = \cos 0 = 1$ (so is not 0). The above says that we can write x as a function of y near (1,0). It actually says a little more: we have $t = \sin^{-1} y$ (as written, the principal inverse-sine), and then

$$x = \cos t = \cos(\sin^{-1} y) = (1 - y^2)^{\frac{1}{2}},$$

which we already knew, I hope. *The conceptual point to grasp is that the general theory tells you something similar is holding, even when one is unable to explicitly eliminate the parameter.*

As an example, consider next the family of curves given in polar coordinates by

$$r = 1 + C\cos\theta, \tag{2}$$

with C varying over the interval $[1, 2]$. You should know that these all admit parametrizations in the same manner, as do *all* polar curves: letting $t = \theta$, we can write the parametric equations [for x and y] that follow from (2):

$$x = r\cos\theta = (1 + C\cos t)\cos t, \qquad y = r\sin t = (1 + C\cos t)\sin t, \tag{3}$$

where $t \in [0, 2\pi]$ traces the curve once.

Hoping it causes no trouble, I'll revert to θ as the variable. We had seen that the curve is a cardioid for $C = 1$, with horizontal cusp (r is non-negative, and $r = 0$ only when $\theta = \pi$). When $C = 2$, the polar variable r *does* take negative values, causing a loop to form in the curve; indeed, this is the case whenever $C > 1$: we solve the inequality $r = 1+C\cos\theta < 0$ (it may help to sketch the graph of cosine) and obtain $\cos\theta < -1/C$. This is equivalent to $\theta \in [\pi - \cos^{-1}(1/C), \pi + \cos^{-1}(1/C)]$. *Note that this interval gets smaller and smaller as* $C \to 1^+$. Its length is $2\cos^{-1}(1/C)$, and $\cos^{-1}(1/C)$ approaches $\cos^{-1}(1) = 0$ as $C \to 1^+$, as does the size of the loop. We can see that the curve passes through the origin for the endpoints of this interval, and thus (a general feature of polar curves) the lines $\theta = \pi - \cos^{-1}(1/C)$ and $\theta = \pi + \cos^{-1}(1/C)$ are tangent to the two branches of the curve through the origin.

If we think in terms of the general considerations from the first page, we have here a collection of examples of parametrized curves containing ones that cross themselves in arbitrarily small time. Thus, the notion of "small enough" depends on the curve: there is no one size of interval I_0 for which one of the variables is a function of the other in all cases. [Does anyone see the ϵ-δ here?] It's also not hard to see that in the examples above, as $C \to 1^+$, the two tangent lines symmetrically approach the horizontal direction. Note also that the peak of the loop (which occurs when $\theta = \pi$) is $C - 1$ units to the right of the origin, and that goes to 0 too.

6. The equations in (4) actually illustrate a point from 15.3, one that will come back to haunt us in 17.6. Let me change the variable C to something more usual, namely s. Then (3) becomes

$$x = (1 + s\cos t)\cos t, \qquad y = (1 + s\cos t)\sin t. \tag{4}$$

To take $\partial x/\partial t$, one can apply (15.3.7) to get

$$\frac{\partial x}{\partial t} = -(1 + s\cos t)\sin t - s\sin t\cos t = \; .... \tag{5}$$

One computes $\partial y/\partial t$ similarly. These give the components of the velocity vectors of the family of curves given in (4), where s is viewed as the extra parameter (constant, but arbitrary).

But we can also elect to hold t constant in (4). Then, for each fixed t, the equations (4) happen in this example to parametrize a *line* in the plane, passing through the point $(\cos t, \sin t)$ and having direction vector $\cos^2 t\,\mathbf{i} + \cos t\sin t\,\mathbf{j}$. We can take $\partial x/\partial s$ and $\partial y/\partial s$, again using (15.3.7). We have a family of lines generated as t varies, with s as parameter on each line, and direction vector varying as above. Try to draw both families of curves in a single picture.