

On the outcome of Exam 2

I wanted to give my comments on the two problems I graded (#8 and #9):

8. The region Ω in question is the triangle with vertices $(0, 0)$, $(-1, 1)$ and $(1, 1)$. There are a few ways to proceed with part a). The most direct is to describe Ω by the inequalities: $0 \leq y \leq 1$; for each y , $-y \leq x \leq y$ (you might display the cross-sections for guidance). This produces the iterated integral

$$(1) \quad \int_0^1 \int_{-y}^y x^2 y^2 dx dy.$$

Some had

$$(2) \quad \int_0^1 \int_{-1}^1 \dots,$$

which is fundamentally wrong (zero credit, as announced); while it's true that $0 \leq y \leq 1$ and $-1 \leq x \leq 1$ for all points of Ω , there are also points outside Ω for which the same inequalities hold. **The student must come to recognize that (2) appears in an (x, y) -integral only for a rectangle.**

For b), it was basically one point for $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$, two for $0 \leq r \leq \frac{1}{\sin \theta}$ ($r = \frac{1}{\sin \theta}$ is $y = 1$ in polar coordinates, and one goes from the origin to the line in each direction), and two for the Jacobian r . Note the *irrelevance* of the range of the function $\frac{1}{\sin \theta}$ on the interval $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$. The correct answer was therefore:

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{\frac{1}{\sin \theta}} (r \cos \theta)^2 (r \sin \theta)^2 r dr d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{\frac{1}{\sin \theta}} r^5 \cos^2 \theta \sin^2 \theta dr d\theta.$$

In c), the Jacobian equals 2. The region Γ in the (u, v) -plane is the triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$. Leaving the generic " Γ " instead of writing an iterated integral (as in (1), but in terms of u and v) got a subtraction of two points. The answer:

$$\int_0^1 \int_0^{1-v} 2(u-v)^2 (u+v)^2 du dv \quad [\text{or equivalent}].$$

You should make sure that when you write an iterated integral, it is in correct general format, I mean for the limits (i.e., endpoints) of integration! If the integrand is (say) $f(u, v) du dv$, the integral should be of the form

$$(3) \quad \int_a^b \int_{c(v)}^{d(v)} f(u, v) du dv,$$

where a and b are constants; as indicated, c and d are functions of v (possibly constant), and both are constant if and only if the region of integration is a rectangle. Among other things, there should not be u 's on the inner integral sign, nor any u 's or v 's on the outer integral sign. Nor should one have any other variables (like x and y) from a previous incarnation.

9. Ah, #9 I felt that the exam needed a question concerning differentiable functions. Recall that it was added while you were taking the exam that, yes, those expressions in (r, θ) *do* define functions on the whole (x, y) -plane. One challenges this assertion at one's own risk. The functions are even continuous.¹

If a student were asked about differentiating some function with respect to (x, y) , I would bet that the first preference would be to have a formula for the function in terms of (x, y) . In a), the expression $r \cos \theta$ is a familiar one, and it equals x , as most students wrote down. This is a formula in terms of (x, y) , a rather easy one. This is as easy an example of an everywhere differentiable function as there is. But there was more confusion than delight with most students at this point.

In b), the function gets expressed as

$$\begin{cases} \frac{x^2}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

This is differentiable everywhere except possibly at the origin. There are various ways to see this function is *not* differentiable at the origin. For one, restrict to the x -axis. Don't be too casual, though, since

$$\begin{cases} \frac{x^3}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is differentiable at the origin (with derivative $\mathbf{0}$).

¹You should know that the relation between continuity and differentiability is a one-way street. Differentiable functions are continuous, but you have simple counterexamples (from Calc I, even) that kill the converse.