

The end of Monday's lecture:

Recall the following:

Let $\mathbf{r} = \mathbf{r}(t)$ be the radial vector of an object moving under a central force, and let $\mathbf{L} = \mathbf{r} \times \mathbf{r}'$. We saw that \mathbf{L} is a constant vector, and the object moves in the plane (through the origin) $\mathbf{L} \bullet \mathbf{r} = 0$. We were on our way to seeing that inside that plane, the rate (with respect to time) that area is swept out by the object is also constant.

This is a question about polar coordinates. The rate at which area is swept out with respect to polar angle θ is given by the Calc II formula, $\frac{1}{2}r^2$, where r is, equivalently, the polar radius or $\|\mathbf{r}\|$ (apply Theorem 5.3.5 to (9.5.1)). Thus if A denotes area swept,

$$A' = \frac{dA}{dt} = \frac{1}{2}r^2 \frac{d\theta}{dt}.$$

We return to $\mathbf{L} = \mathbf{r} \times \mathbf{r}'$. Let $\mathbf{u}_r = \frac{\mathbf{r}}{r}$. Then $\mathbf{r} = r\mathbf{u}_r$, and $\mathbf{r}' = r'\mathbf{u}_r + r(\mathbf{u}_r)'$. This gives

$$(*) \quad \mathbf{L} = r\mathbf{u}_r \times r(\mathbf{u}_r)' = r^2\mathbf{u}_r \times (\mathbf{u}_r)'.$$

It is time to get explicit. In polar coordinates, $\mathbf{u}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$. Then

$$(\mathbf{u}_r)' = \frac{d\mathbf{u}_r}{d\theta} \frac{d\theta}{dt} = \mathbf{u}_\theta \frac{d\theta}{dt},$$

with $\mathbf{u}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}$. Putting this into (*), we get:

$$\mathbf{L} = r^2\mathbf{u}_r \times \mathbf{u}_\theta \frac{d\theta}{dt}.$$

Note that \mathbf{u}_r and \mathbf{u}_θ are perpendicular unit vectors. It follows that $\frac{1}{2}\|\mathbf{L}\| = \frac{1}{2}r^2 \frac{d\theta}{dt} = A'$, and this is indeed a constant.