

Antiderivatives, differentials and gradients

In 15.9, there are two new general concepts involving subsets of the plane. We will not harp on their rigorous, abstract definitions in this course, for an intuitive grasp is both sufficient and possible. Before mentioning them, we recall that an open set is said to be *connected* (page 880) if it comes in one piece. As illustrations in one variable, $(1, 3) \cup (2, 4) = (1, 4)$ is connected, whereas $(1, 2) \cup (2, 4)$ is not.

The first new notion is that of a *simple closed curve*. It is nicely stated in the middle of page 934, in terms of any parametrization of the curve. Basically, it says that the curve doesn't intersect itself before it closes up. The other is: a connected open set (a.k.a. a *region*) in the plane is said to be *simply connected* if there are no holes in it. Here, a hole can be either a region or just a point. Examples of regions that are not simply connected are:

$$(*) \quad \{(x, y) : 1 < x^2 + y^2 < 2\} \quad \text{and} \quad \{(x, y) : 0 < x^2 + y^2 < 2\}.$$

A clearer mathematical statement is that a region Ω in the plane is simply connected whenever for every simple closed curve contained in Ω , the region it bounds is also contained in Ω . (Do we truly understand what we meant by “hole”?) In both examples in (*), the unit circle is a simple closed curve whose region it bounds (the unit disc) includes the hole and thus fails to be contained as a subset of the region.

In drawing general pictures, we often make our curves wiggly to demonstrate to ourselves and others that issues of symmetry and the like are irrelevant.

In Calc I, we had the fundamental theorem (Thm. 5.3.5): *If $f(x)$ is a continuous function on an open interval I , there is a differentiable function $g(x)$ defined on I for which $dg = f(x)dx$ (i.e., $g'(x) = f(x)$).* We are treating now the Calc III analogue of this fact.

It's more complicated, but still manageable. Let $P(x, y)$ and $Q(x, y)$ be continuous functions defined on some open region Ω in the plane. At issue is whether we can find a differentiable function $g(x, y)$, also defined in Ω ,—surely, we don't expect g to have larger domain if P and Q “have problems” outside Ω —such that

$$(1) \quad dg = P(x, y)dx + Q(x, y)dy.$$

Note that this is just asking to determine those g for which

$$(2) \quad g_x = P \quad \text{and} \quad g_y = Q$$

identically in Ω . We can see without strain that it is equivalent to have

$$(3) \quad \nabla g = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j};$$

indeed, it is common to view dg and ∇g as two incarnations of the same thing. We know (Thm. 15.3.3) that on a connected open set, any two anti-derivatives differ by a constant (just as for functions of one variable in Calc I).

Because differentiability in two or more variables is a slightly annoying concept, we place ourselves closer to “in practice, ...” by assuming that P and Q have continuous partials of first order, as in Thm. 15.1.3, i.e., that P and Q are *continuously differentiable*, so that we may invoke the fact about mixed partials from 14.6. From the latter, we know that in order to have a chance of finding g as in (2), we need to have the equation of compatibility

$$(4) \quad P_y(x, y) = Q_x(x, y)$$

(that follows from $g_{xy} = g_{yx}$) hold in Ω . Here is a reasonable way of stating the conclusion in this situation.

Theorem. Let P and Q be defined on a region D in the plane, with (4) holding at every point of D . Then for every simply connected open subset Ω of D , there is a function g defined on Ω such that $dg = Pdx + Qdy$ on Ω .

Since discs are simply connected, the above Theorem allows for the fact that something (having an anti-derivative, in the present case) might hold only *locally*, i.e., on small open sets, but not *globally*, i.e., on the whole domain. If you follow this document, you will understand how that is possible. In addition, the method of partial integration from 15.9, which is Calc I with parameter, might be viewed in these terms.

For the rest of this document, let

$$(5) \quad P(x, y) = \frac{-y}{x^2 + y^2}, \quad Q(x, y) = \frac{x}{x^2 + y^2}.$$

With a little care, we can verify that (4) above is satisfied by the functions in (5). With a little eyesight, we can see that these functions are not defined when (and only when) $(x, y) = (0, 0)$. The default domain D of both P and Q is the (x, y) -plane with the origin deleted. It is not simply connected, as it has a hole at the origin. **This will turn out to be crucial.** However, the subset $\Omega = \{(x, y) : x < 0 \text{ when } y = 0\}$, where a whole *ray* has been deleted from the plane, *is* simply connected (draw a picture).

Since anti-derivatives all differ by a constant (Thm. 15.3.3 again), we proceed with the observation that $g = \theta$ (the polar angle) gives the solution in Ω as in the theorem above. To be more precise, we take the expression

$$g(x, y) = \tan^{-1} \left(\frac{y}{x} \right),$$

and simply take its differential. (What *does* that mean when $x = 0$?) First

$$\frac{\partial}{\partial y} \tan^{-1} \left(\frac{y}{x} \right) = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x},$$

which equals $Q(x, y)$, and I leave it to you to verify or accept the fact that

$$\frac{\partial}{\partial x} \tan^{-1} \left(\frac{y}{x} \right) = P(x, y).$$

The point to grasp is that θ —with values in $(0, 2\pi)$ —is a differentiable function on Ω , but there is not even a *continuous* function that one might call θ on all of D : it is Calc 0 that when we go once around the origin, e.g., counterclockwise around the unit circle (a simple closed curve contained in D), you add on 2π to the polar angle. The same holds for the circle of any radius centered at the origin. (Yes, that's independent of r , as it must be by Thm. 15.3.3.)

I will write out very carefully the reason that there is no function g , defined on all of D above, with $dg = Pdx + Qdy$ on D . First, it is not enough to say that D isn't simply connected, for that is (silly) “inverse reasoning” from the theorem at the top of this page (see *Implications in mathematics*). After all, the function $g = 0$ (or any constant function) solves the anti-derivative problem for $P = Q = 0$

on any domain in the plane, simply connected or not. What is true that when D is not simply connected, *there exists* a pair of functions P and Q with continuous partials, satisfying $Q_x = P_y$, for which no such g exists. (Next week's material may give some insight as to why.)

In the case at hand, we have alleged that the functions in (5) give such a pair: on the subdomain Ω , θ is one solution for g ; all such are of the form $\theta + C$, where C is a constant. Since constant functions are defined everywhere, we may as well take $C = 0$. We thus seek a differentiable function defined on D which agrees with θ on Ω . It has to be near 0 at points a little above the positive x -axis, and near 2π at points a little below. But we can see that such a function would be **discontinuous** along the positive x -axis, so is *a fortiori* not differentiable there (15.1.6).