

Fundamentals

We have been talking primarily about the “cheap” version of the Fundamental Theorem of Calculus. What about the “chunky” one? In Calc I, the latter was:

Theorem (FTC-I). *Let $P(x)$ be a continuous function on the open interval I , and let a be any point of I . Then*

$$g(x) = \int_a^x P(t)dt$$

is an antiderivative of $P(x)$; that is, $dg = Pdx$.

(This is pretty obvious if you understand the basic properties of the integral.) We note that there are no exciting line integrals in one variable.

To state the multivariate version of this (for simplicity of notation only, we'll write things down for 2 variables only), we take as given a differential $Pdx + Qdy$, with P and Q continuous functions on some open set Ω of the (x, y) -plane, and view it as an integrand for line integrals over all piecewise smooth curves C in Ω . Recall that we said that the line integral

$$(*) \quad F(C) = \int_C Pdx + Qdy$$

is *independent of path* if $F(C) = F(C')$ whenever C and C' are curves in Ω having the same initial point \mathbf{A} and terminal point \mathbf{B} , i.e., when the value of $F(C)$ is determined by the points \mathbf{A} and \mathbf{B} ; the detail of *how* the curve gets from \mathbf{A} to \mathbf{B} (within Ω) is irrelevant.

We have already seen the “cheap” version of the FTC:

Theorem 17.2.1. *If $Pdx + Qdy = dg$ for some function g , then $\int_C Pdx + Qdy$ is independent of path.*

(Indeed, the value of above integral is then just $g(\mathbf{B}) - g(\mathbf{A})$.) The point of this document is verifying the converse, which is the analogue of FTC-I:

Theorem (FTC-III). *If the line integral $(*)$ is independent of path, then there is a function g on Ω for which $dg = Pdx + Qdy$.*

Proof. Let \mathbf{A} be any one point of Ω . (It plays the role of a in FTC-I.) Let \mathbf{B} denote an arbitrary point of Ω (playing the role of x in FTC-I). Define a function g on Ω by the formula:

$$g(\mathbf{B}) = \int_C Pdx + Qdy.$$

where C is any curve in Ω from \mathbf{A} to \mathbf{B} . By assumption, this gives the same number, whichever C one takes, so it really defines a function of \mathbf{B} , as written. We may as well write, then, that

$$g(\mathbf{B}) = \int_{\mathbf{A}}^{\mathbf{B}} Pdx + Qdy.$$

We show that

$$dg = Pdx + Qdy.$$

We'll see that this follows more or less directly from FTC-I. Let's show that $g_x = P$ (that $g_y = Q$ is similar). The basic picture is: If C goes from \mathbf{A} to \mathbf{B} in Ω , and $|t|$ is sufficiently small, then $\mathbf{B} + t\mathbf{i}$ is in Ω , and can be connected to \mathbf{B} by a horizontal straight line segment (then to \mathbf{A} by $-C$). Of course y is constant on that line segment, and we have a one-variable problem. This brings us to the point where we are effectively in the same situation as in FTC-I.

We can now write

$$f(t) = g(\mathbf{B} + t\mathbf{i}) - g(\mathbf{B}) = \int_0^{t\mathbf{i}} Pdx.$$

Can you now see that $f'(0) = g_x(\mathbf{B}) = P(\mathbf{B})$?