Double trouble?

I was asked about 16.4 #11. I promised to post this, but I did have second thoughts (see #4 below). This is about determining the inequalities defining a region in polar coordinates. The way it is done is basically the same as determining the inequalities for rectangular coordinates. To be specific:

1. The region in question is $\Omega = \{(x, y) : \frac{1}{2} \leq x \leq 1,\ 0 \leq y \leq \sqrt{1 - x^2}\}$.

2. If we take this at face value—do we ever do otherwise?—we see we are taking the intersection of the strip $\frac{1}{2} \leq x \leq 1$ with the upper half of the unit disc.

3. [Draw a picture!] The region Ω can also be described as the one bounded by the three curves: the portion of the unit circle in the first quadrant, the line $x = \frac{1}{2}$ and the line $y = 0$. The first two intersect in the point $P = (1/2, \sqrt{3}/2)$.

4. We want to set up inequalities for the polar coordinates, *familiar quantities from Calc II*. At this point, the student who's a little weak in polar coordinates should look at that material again; it's in Ch. 9, and area in polar coordinates is specifically treated in 9.5. **Shame on you if you needed to see this section again, but didn't refer back to it!** (See item #4 of the academic orientation document, *To the Freshmen*, which is part of the reading material of this course.)

5. It goes more or less as in #1 above. It is "usually" right to specify an interval for θ, and then an interval for r that depends on θ. The θ-coordinate of the point P is $\pi/3$. Thus, in the region Ω, $0 \leq \theta \leq \pi/3$. For each such θ, r goes "from the vertical line $x = \frac{1}{2}$ to the unit circle," which we understand to mean "... radially."

6. The polar equation of the unit circle is $r = 1$. That's easy. The polar equation of the line $x = \frac{1}{2}$ is found by simply converting the latter to polar coordinates: $r \cos\theta = \frac{1}{2}$, i.e.,

$$r = \frac{1}{2\cos\theta}\ ,$$

which we may write if we wish: $r = \frac{1}{2}\sec\theta$. In other words, in polar coordinates Ω is given as:

$$\{(r, \theta) : 0 \leq \theta \leq \tfrac{\pi}{3},\ \tfrac{1}{2}\sec\theta \leq r \leq 1\}.$$

7. Isn't this what we mean by *changing variables* (in this case, from (x, y) to (r, θ))? For integration, we need only remember to replace "$dydx$" by "$\underline{r}drd\theta$."