110.607 Complex Variables, Spring 2004:

Final Exam (take-home, due Monday May 3, 5:00pm)

1. From p. 176 of the text: #1, 4.

2. From p. 194 of text: #1,3,5.

3. Let D be the unit disc, $\overline{D}$ its closure. Let G be a simply connected bounded domain in $\mathbb{C}$, and $\overline{G}$ the closure of G.

a) Show that every analytic automorphism of D extends continuously to $\overline{D}$.

b) Either carry out the following argument, or point out the fallacy:
Fix any point a in G. Then

$$\mathcal{G} = \{g : \overline{G} \to \overline{D} \mid g \text{ is continuous, 1-1, with } g(a) = 0,\ g'(a) > 0,\ g|_G \text{ analytic}\}$$

is a (non-empty) normal family of functions on $\overline{G}$. Find $g \in \mathcal{G}$ that maximizes $g'(a)$. The Riemann mapping $(G, a) \to (D, 0)$ (this is standard notation for a mapping of pairs) extends to the continuous mapping $g : (\overline{G}, a) \to (\overline{D}, 0)$.

4. We say an analytic automorphism T of $\mathbf{P}^1$ (the Riemann sphere) is *circle-preserving* if every circle in the plane, or "circle" through the point ∞ (i.e., closure of a straight line in the plane), is taken (as a set) by T to a circle in the preceding sense.

a) Show that a Möbius transformation is circle-preserving.

b) Show that, given three distinct complex numbers α, β and γ, there is a unique Möbius transformation M with $M(\alpha) = 0$, $M(\beta) = 1$, $M(\gamma) = \infty$.

c) Is the composition of two circle-preserving automorphisms circle-preserving? *Explain.*

d) Show that a circle-preserving T that fixes three points is the identity.

e) Conclude that a circle-preserving T must be a Möbius transformation.

f) What happens if we remove the assumption that T be analytic?

5. *(Bonus)* Prove the Riemann Hypothesis (p.193). ;)

Solutions

1. #1. With so many identities between sine and cosine, you'd think that we could deduce this from the product expansion of $\sin \pi z$. It seems simplest to work it out from scratch, following the example of $\sin \pi z$.

The zeros of $\cos \pi z$ come in pairs of the form $\pm(n - \frac{1}{2})$ $(n = 0, 1, 2, ...)$, and they are all simple zeros. The basic elementary factors are

$$\left(1 \pm \frac{z}{n - \frac{1}{2}}\right) = \left(1 \pm \frac{2z}{2n - 1}\right).$$

Taken as a pair for each n, they give the factor

$$1 - \frac{4z^2}{(2n-1)^2}.$$

As with $\sin \pi z$, the product

$$P(z) = \prod_{n=0}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right)$$

converges absolutely, etc., to an entire function with the same zeros as $\cos \pi z$; no additional convergence factors are required. It follows that

$$\cos \pi z = e^{f(z)} P(z)$$

for some entire function f. It is evident that $f(0) = 0$; we must show that $f(z) \equiv 0$. What should we do, imitate the argument for $\sin \pi z$?

Of course, that works, but J.P. found it wise to invoke from the start the *multiplicative* identity

$$\cos x = \frac{\sin(2x)}{2 \sin x},$$

which leads to a nice solution.

#4. Now where might that identity come from? Take the expression as given and it's weird; take the reciprocal, and it comes from the product expansion of $\sin z$, for $z = \frac{1}{2}$.

2. #1. Go for it:

$$\xi(z) = z(z-1)\pi^{-z/2}\zeta(z)\Gamma(\tfrac{z}{2}).$$

We need to show first that ξ has only removable singularities. Well, $\zeta(z)$ has a simple pole at $z = 1$ and a simple zero at $z = -2, -4, -6, \ldots$, and $\Gamma(z)$ has simple poles at all non-positive integers; $\Gamma(\frac{z}{2})$ has poles at all non-positive *even* integers. These all cancel except for the poles at $z = 0$ and $z = 1$. Those first two factors take care of that. Thus ξ is entire.

Now, write down

$$\xi(1-z) = (1-z)(-z)\pi^{-(1-z)/2}\zeta(1-z)\Gamma(\tfrac{1-z}{2}).$$

It seems like the next step is inserting the functional equation of $\zeta(z)$:

$$\zeta(1-z) = \frac{\zeta(z)}{2(2\pi)^{z-1}\Gamma(1-z)\sin(\frac{\pi z}{2})}.$$

When you do that, it seems like a new functional equation for the Gamma function is needed ...., or is there a misprint in Conway? ....

#3. You should sense the need to use the Euler product formula for $\zeta(z)$. With that said, we know that $\zeta^2(z)$ has factors of the form

$$\left(\frac{1}{1-p^{-z}}\right)^2 = (1 + p^{-z} + (p^{-z})^2 + \ldots)^2.$$

For the present purpose (combinatorial), it does not matter is we suppress the exponent "$-z$" in the above; i.e., consider

$$\left(\frac{1}{1-p}\right)^2 = (1 + p + p^2 + p^3 \ldots)^2.$$

We claim that the coefficient of p^m equals $m + 1$, the number of divisors of p^m. This is not difficult; you may remember it from Calculus II.

It's only slightly more difficult that the number of divisors of a finite product of p^m's for distinct p is the product of the $m + 1$'s. That is, the number of divisors is multiplicative with respect to the primes. The statement follows.

#5. The Euler factor here is

(*)

$$(1 - p^{-z})((1 + p^{-z+1} + (p^{-z+1})^2 + \dots) = (1 - p^{-z})((1 + pp^{-z} + p^2(p^{-z})^2 + \dots)$$
$$= 1 + (p - 1)p^{-z} + (p^2 - p)(p^{-z})^2 + \dots).$$

One checks that $\phi(n)$ is multiplicative for relatively prime integers, and that $\phi(p^m)$ is given by the coefficients of the right-hand member of (*). By the way, $\phi(n)$ is a well-known function, due to Euler; you've probably seen it before in Algebra.

3. a) We have determined all automorphisms of D. They are of the form

$$T(z) = u\frac{z - \alpha}{1 - \overline{\alpha}z},$$

where $|u| = 1$ and $|\alpha| < 1$. We see that T, as a function on the plane, has one singularity, namely at $z = \overline{\alpha}^{-1}$. This point is not in $\overline{D}$, so T is a differentiable function on $\overline{D}$. (The image is $\overline{D}$ as well.)

b) The key point is to decide whether $\mathcal{G}$ is a normal family. It is a good idea to try the case $G = D$. Exploring, consider

$$g_\alpha(z) = \frac{z - \alpha}{1 - \alpha z},$$

for α *real* and approaching 1 from below. (We can take a sequence of such α if you wish.) The question is whether we can extract a subsequence that is uniformly convergent on $\overline{D}$ (yes, $\overline{D}$). Since

$$g_\alpha(z) \to -1$$

pointwise for $z \in \overline{D} - \{1\}$, the uniform limit, necessarily continuous, would have to be the constant function -1. On the other hand, $g_\alpha(1) = 1$ for all $\alpha \neq 1$. There is no uniform limit, and we can stop here.

4. a) I like this argument: If $T(z) = az + b$ (equivalently $T(\infty) = \infty$) or $T(z) = S_a(z) = (z - a)^{-1}$, we see easily that T is circle preserving. We want to reduce the general case to these.

Let $a = T^{-1}(\infty)$. Then $R = S_a T^{-1}$ fixes the point ∞, and $T = R^{-1}S_a$. We'll invoke the *really* easy part c) to finish.

b) Write the general Möbius transformation as

$$M^{-1}(z) = \frac{az + b}{cz + d},$$

this way for convenience. Thus, the conditions are $M^{-1}(0) = \alpha$, $M^{-1}(1) = \beta$, and $M^{-1}(\infty) = \gamma$. (One can even allow one of the numbers to be ∞.) Explicitly, that's

$$\frac{b}{d} = \alpha, \quad \frac{a+b}{c+d} = \beta, \quad \frac{a}{c} = \gamma.$$

or, $b = \alpha d$, $a = \gamma c$, $a + b = \beta(c + d)$. Now simply solve these equations. The solution for (a, b, c, d) is unique up to a scalar, so M^{-1} and then M is uniquely determined.

c) was covered in conjunction with a), so we pass onto

d) Because of b), we can arrange that the three points are 0, 1, ∞. By circle-preservation, T takes the real axis $y = 0$ to itself. The "circles" $y = B$ const are all disjoint (outside ∞), and are taken by T to lines that do not intersect each other in the plane. Thus, each $y = B$ is taken to some $y = B'$. Similarly, the family of vertical lines $x = A$ goes to some family of parallel lines $x = my + D$. *Note that we've used nothing yet about the differentiability of* T.

Suppose first only that T is differentiable. Write $T = u + iv$ as usual. The families of lines yield that the partial derivatives are of the form: $u_x = f(x, y)$, $v_x = 0$ $[v = \phi(y)]$, $u_y = mg(x, y)$, $v_y = g(x, y)$. Keep this for later.

When we assume T to be complex analytic, the Cauchy-Riemann equations lead to $m = 0$ and $f = g$. Since $u_y = 0$, we write $u = \psi(x)$; then CR's $u_x = v_y$ gives $\psi'(x) = \phi'(y)$, so both derivatives are the same constant. Thus we have $u = Ex+F$ and $v = Ey+G$, where E, F, G are real constants; and $T(0) = 0$ gives $F = G = 0$. So $u = Ex$ and $v = Ey$. Then $E = 1$, because of the behavior of T on the x-axis, and we get that T is the identity: $T(z) = z$.

e) Given circle-preserving T, let $\alpha = T(0)$, $\beta = T(1)$, $\gamma = T(\infty)$. Let, M be, as in b), the Möbius transformation with $M(\alpha) = 0$, $M(\beta) = 1$, $M(\gamma) = \infty$. Then MT fixes 0, 1, and ∞. By c), MT is circle-preserving, so by d) it is the identity. Thus, $T = M^{-1}$ is a Möbius transformation.

f) The most obvious thing that happens is that we must allow complex conjugation. $T(z) = \overline{z}$ as a circle-preserving mapping that fixes 0, 1, ∞. But this is not so far from being holomorphic, you know. Then we would have that all conjugate-Möbius transformations are circle-preserving. This, however, is barely getting into the question.

It is enough to determine all differentiable circle-preserving T that fix the three points. Go back to $u_x = f(x, y)$, $v = \phi(y)$, $u_y = mg(x, y)$, $v_y = g(x, y)$. So $g(x, y) = \phi'(y)$, $u_y = m\phi'(y)$, and thus:

$$u = D(x) + m\phi(y), \quad \text{and} \quad v = \phi(y).$$

Does anyone want to take this further? (It might be a good idea to look at some additional circles.)